Motivation: JT gravity

GR is difficult to understand as a quantum theory

≈ simpler: lower dimension

Let's try in 2d:

Naively, \[ I = \sum \int d^2 x \sqrt{g} \; R \]

Ricci scalar

Riemann metric

Surface

However: purely topological (Gauß-Bonnet)

≈ add a (real-valued) scalar \( \phi \)

\[ I = \frac{1}{\kappa} \sum \int d^3 x \sqrt{g} \cdot \phi \cdot (R + 2) \] JT gravity

Classical solution: \( R = -2 \)

\( \Sigma \) is hyperbolic Riemann surface

Feynman path integral:

\[ Z_\Sigma = \frac{1}{\text{vol}} \int \mathcal{D}\phi \; \mathcal{D}g \; \exp \left( -\frac{1}{\kappa} \sum \int d^3 x \sqrt{g} (R + 2) \phi \right) \]

\( \phi \)

\( \mathcal{D}\phi \)

\( \sqrt{g} \)

\( \mathcal{D}g \)

Volume of diffeomorphism

Integrate over \( \phi \) \( \Rightarrow \mathcal{D}(R + 2) \)

⇒ integral localizes on \( \mathcal{M}_g = \text{moduli space of } 2\text{-mfld's with hyperbolic structure} \)

\( \Rightarrow Z(\Sigma) = \int \mathcal{X}(\Sigma) \rho \; \mathcal{D}_\mu \)

\( \rho \)

\( \mathcal{X}(\Sigma) \)
\[ \chi(\Sigma) = \frac{C \chi(\Sigma)}{\text{vol}(\mathcal{M}_g)} \]

\[ \chi(\Sigma) = 2g-2 \]

There is a similar story if \( \Sigma \) has boundary components of prescribed lengths. Again:

\[ \chi(\Sigma_{g,b}) \sim \text{vol}(\mathcal{M}_{g,b}) \]

(see e.g. Witten @ WHCGP)

In the next few lectures, we will describe Mirzakhani's work on these symplectic volumes:

2 ways:

1. via symplectic reduction & intersection theory of tautological classes (Today + next time)

M. Mirzakhani '06

2. via Moschane identities & recursion

(Charlie & Lewis)

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The moduli space \( \mathcal{M}_{g,n} \)

Let \( S = \Sigma_{g,n} \) be an oriented, smooth surface

\[ \chi(S) = 2 - 2g - n < 0 \]

i.e. \( g = 0 \Rightarrow n \geq 3 \)

\[ n = 1 \Rightarrow n \geq 1 \]
\[ \begin{align*}
\text{i.e. } \quad g=0 & \Rightarrow n \geq 3 \\
g=1 & \Rightarrow n \geq 1
\end{align*} \]

**Def.** The Teichmüller space \( \mathcal{T}(S) \) is

\[ \mathcal{T}(S) = \left\{ \begin{array}{l}
\text{\ : complete hyp. surface} \\
\text{\ : diff. } f : S \to X
\end{array} \right\} \sim \]

where \((X, f) \sim (Y, g)\) if \(f \circ g^{-1} : Y \to X\) is isotopic to a conformal map.

If \(\partial S \neq \emptyset\ (n+0)\), \(\text{Fix } b = (b_i) \in \mathbb{R}_+^n\)

\[ \mathcal{T}(S, \bar{b}) = \left\{ \begin{array}{l}
\text{\ : complete hyp. surface with geodesic boundary} \\
\text{\ : } f : S \to X \\
\text{\ s.t. } \ell_\beta(x) = b_\beta \\
\text{\ : } (\beta = 1, \ldots, n)
\end{array} \right\} \sim \]

We also write \( \mathcal{T}_{\bar{b}}(S^n) = \mathcal{T}_{\bar{b}}(S^n) = \mathcal{T}(S^n, \bar{b}) \)

The mapping class group
\[ \text{Mod}_{S^n} := \text{Mod}(S^n) := \left\{ \begin{array}{l}
\text{isotopy classes of orientation-pres.} \\
\text{homeo's } S \to S \text{ fix all bdry}
\end{array} \right\} \]

comp. setwise
Mod\(_{g_n}\) \(\cong\) \(\mathcal{J}_{g_0, b}^n\) by \(\phi_{b}(x, t) := (x, t \cdot \phi)\)

\(\sim\) quotient

\(\mathcal{M}_{g, b} := \mathcal{M}_{g, (b')} := \mathcal{M}(S_{g+n}, b') = \mathcal{J}_{g, b}/\text{Mod}_{g_n}\)

Moduli space of Riem. surfaces

**Important features:**

- \(\mathcal{J}_{g_b}\) has nice coordinates

**Fenchel-Nielsen coordinates**

Fix a pair-of-pants decomposition \(P\) of \(S_{g+n}\)

- get \(3g - 3 + n\) curves

each of them has a unique geodesic representative \(\alpha_i\) in its homotopy class

Fenchel-Nielsen coords:

\[\left\{ \ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_k}(X) \right\}\]

(length of \(\alpha_i\); twisting parameter)

\(\sim\) give an iso

\[\mathcal{J}_{g_b} \sim \mathbb{R}_+^k \times \mathbb{R}^k\]

- \(\mathcal{J}_{g_b}\) has a natural symplectic form

  West-Peterson form

\[\omega = \omega_{WP}\]

(\(\text{Goldman}\))

that is \(\text{Mod}_{g_n}\) -invariant

- \(\mathcal{M}_{g_b}\) has a symplectic form \(\omega\)

& we can compute \(\text{vol}(\mathcal{M}_{g_b}) = \int e^\omega\)

In FN coords:

\[\omega = \sum_{i=1}^{k} \ell_{\alpha_i} \, d\tau_i \wedge d\ell_{\alpha_i}\]
In FN-coords:  \[ \omega_{wp} = \frac{k}{i=1} dx_i \wedge d\xi_i \]

**Symplectic reduction** (Marsden-Weinstein reduction)

Let \((M, \omega)\) symplectic \& \(G\) compact Lie gp preserves \(\omega\)

Let \(\gamma : \mathfrak{g} \to \text{Vect}(M)\) be the infinitesimal action

**Def:** A moment map for the \(G\)-action on \(M\) is a \(G\)-equiv. map \(\mu : M \to \mathfrak{g}^*\)

\[ \forall x \in \mathfrak{g} : \gamma_x \omega = d\mu_x \]

Where \(\mu_x = \mu \cdot x : M \to \mathbb{R}\)

**Rmk:** Can always shift \(\mu \to \mu + \zeta, \zeta \in \mathfrak{g}^/[\mathfrak{g}, \mathfrak{g}]^*\)

and any two moment maps differ by such a \(\zeta\)

- \(G\) semisimple \(\Rightarrow\) moment unique
- \(G\) abelian \(\Rightarrow\) \(\zeta \in \mathfrak{g}^*\)

**Ex:** \((M, \omega) = (\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i), G = U(1)\)

has a moment map \(\mu(x, y) \cdot \mathbb{R}^\times = -\frac{1}{2} (\|x\|^2 + \|y\|^2)\)

**Thm:** Suppose \((M, \omega, G, \mu)\) is a Hamiltonian \(G\)-space and suppose \(G \cdot \mu^{-1}(0)\) freely.

Thus \(\mathcal{O} \cdot \mu^{-1}(0)\) ...
Then:

1. $\mu^{-1}(0)/G$ is a manifold
2. $\pi: \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal $G$-bundle
3. $\exists!$ sympl form $\omega$ on $\mu^{-1}(0)/G$ s.t.
   
   \[ \pi^*\omega = \pi^*\omega \]

\[ G \xrightarrow{\rho} \mu^{-1}(0) \xrightarrow{\pi} M \]

Ex.: $(M,\omega) = (\mathbb{C}^n, \omega_{FS})$

\[ \sim \mu^{-1}(0)/G \cong \left( \mathbb{C}P^{n-1}, \omega_{FS} \right) \]

Q: How do $(M_a,\omega_a)$ & $(M_b,\omega_b)$ relate for a "close to $b$"

Thm.: [Normal Form Thm]

$G = U(1)^n$

Suppose $G \trianglelefteq \mu^{-1}(0)$ freely. There exists an $\varepsilon > 0$

s.t. $G \trianglelefteq \mu^{-1}(a)$ freely \( \forall a \in B_{\varepsilon}(0) \)

And we have:

\[ M_a \cong M_0 \]

& if you pick a connection $A$ on $\mu^{-1}(0) \to \mu^{-1}(0)/G$,

then

\[ (M_a,\omega_a) \cong (M_0, \omega_0 + a\Omega) \]

sympctomorphic curvature form of $A$
The rule: The \( \mathbb{U}(1)^n \)-bundle is really \( n \) \( \mathbb{U}(1) \)-bundles \( C_i, \ldots, C_n \).

Theorem: Then the symplectic forms are related via
\[
[W_a] = [\omega_0] + \sum_{i=1}^n a_i [\phi_i], \quad \phi_i \in C_i(C)
\]

Corollary: \((M, \omega, \mathbb{U}(1)^n, \mu)\) a Hamiltonian system, \( \mu^{-1}(0) \subseteq \mathbb{U}(1)^n \) freely.

Then for sufficiently small \( \|a\| < \varepsilon \), \( \text{vol}(M_a, \omega_a) \) is a polynomial of degree \( d = \frac{\text{dim } M}{2} \) given by
\[
\text{vol}(M_a, \omega_a) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n, \lvert \alpha \rvert \leq d} C(\alpha) a^\alpha
\]

where \( \alpha! (m-1\alpha)! \), \( C(\alpha) = \int_{M_0} \phi_1^{\alpha_1} \cdots \phi_n^{\alpha_n} \omega \)

Guillemin; Moment maps... '06 ... bordered R.S.'s [8] ?

Theorem: \((M_r, \omega_r)\) symplectic \( r=1,2 \) \( \dim_r = 2n \)

\( \mathbb{Z} \) connected manifold \( \dim k > n \) and \( \psi_r: \mathbb{Z} \to \mathbb{U}_r \)

Coisotropic embeddings

\( s.t. \quad \psi_1^* \omega_1 = \psi_2^* \omega_2 \)
\[ \pi^* \omega_1 = \pi^* \omega_2 \]
Then there exist nbhds \( U_i \) of \( \varphi_i(z) \) & a sympl. \( \phi : U_i \to U_j \)

\[
\omega = \pi^* \omega_0 + d(tA), \quad -\varepsilon < t < \varepsilon
\]
\[\uparrow_{\text{cxn}}\]