"More on Fractons"

GST Seminar — 09/23/2020

Based mostly on 2008.03852 by Kevin Slagle.

Also relevant: 2002.05186 by David Aasen ①, Daniel Bulmash ②, Abhinav Prem, Kevin Slagle ③ and Dominic Williamson.

①, ② and ③ have also given talks on this subject at Quantum Matter/Condensed Matter Seminars at CMSA, Harvard. These talks are available online.

Outline:

1. X-cube model (Review of Andreas' talk)
2. X-cube model from topological defect network (2002.05166)

   (This will be very brief since I could not understand the paper well)
3. TFTs of (2+1) D and (3+1) D toric codes
4. Foliations ; foliated gauge fields ; foliated QFT of X-cube model, its observables and their mobility constraints.

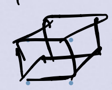
① X-cube model:

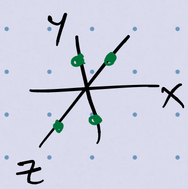
$$H = -\sum_c A_c - \sum_{v,k} B_v^k$$

c : cube

v : vertex

$k = x, y, z$

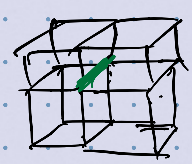
A_c = product of 12 σ^x on cube 

$B_v^x =$ 

σ^z at green dots.

Ground state: $A_c = B_v^k = +1 \quad \forall c, v, k.$

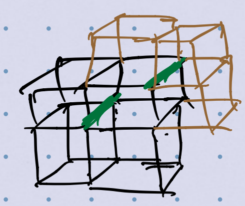
There are $2^{2L_x + 2L_y + 2L_z - 3}$ ground states.

Excitations: i). 

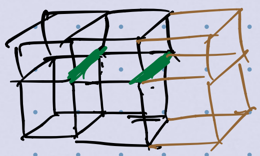
Act by σ^z on green link.
$\Rightarrow$ the four cubes have $A_c = -1$.

These are called fractons.

Moving a fracton gives a config with 6 (instead of 4) fractons.



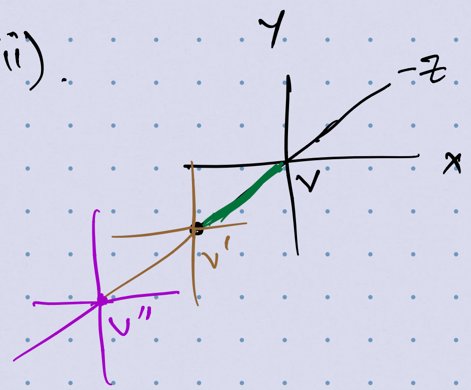
$\Rightarrow$ "Immobile".



Can move two without costing energy.

i.e. a pair of fractons is a "<u>planon</u>".

ii).



There also exist <u>lineons</u> in this model.

Act at green link by σ^x

This creates excitations at v, v' with

$$B_v^x = B_v^y = -1 \quad , \quad B_v^z = +1 .$$

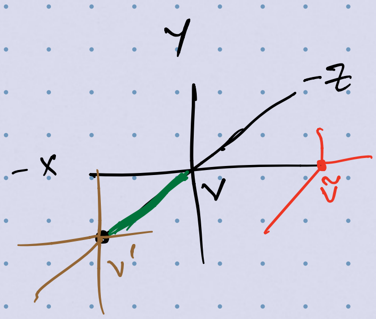
$$B_{v'}^x = B_{v'}^y = -1 \quad , \quad B_{v'}^z = +1$$

- can move along z-axis without costing any energy. by acting on the brown link along z-axis, for example.

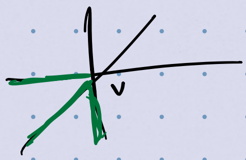
- However, moving along, e.g. x-axis costs energy.

$$B_v^x = -1 , \ B_v^y = B_v^z = +1$$

$$B_{\tilde{v}}^x = +1 , \ B_{\tilde{v}}^y = B_{\tilde{v}}^z = -1 .$$



- Lineons along three different directions annihilate.



e.g. action of σ^x at the three links gives

$$B_v^x = B_v^y = B_v^z = +1$$

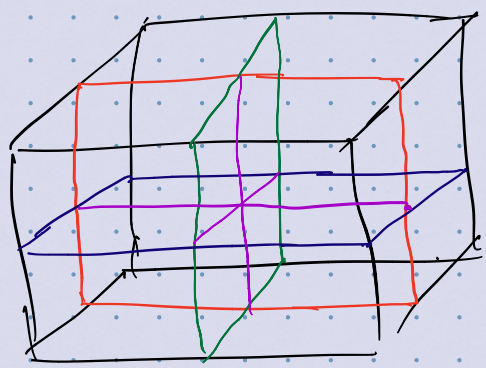
These properties are characteristic of X-cube model and we hope to reproduce them in a QFT in continuum.

(2) X-cube model from topological defect network of 3+1D toric code:

(*)

Aasen - Bulmash - Prem - Slagle - Williamson (2002. 05166)

show that all known gapped fracton models can be constructed through networks of defects in topological QFTs.



Black: A region inside the cubic lattice

Red, Green, Blue: Three orthogonal planes inside the lattice.

Purple: Intersection of two codimension-1 defects.

(*) shows that

X-cube model = 3+1D toric code on cubic lattice

+ (2+1)D toric code on codim-1 defects

+ couplings

I don't understand the details. (It will be nice to understand them).

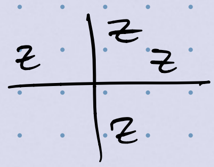
But I want to comment that this inspired the author of 2008.03852 to write down the continuum description of X-cube model on an arbitrary (foliated) manifold.

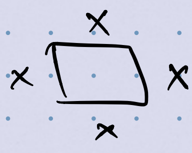
This is, of course, because the TFTs for 2+1 D & 3+1 D toric codes are very well known.

In what follows, we'll review them first and then discuss the foliated QFT of X-cube model.

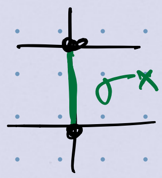
(3) TFT of (2+1)D and (3+1)D toric codes:

2+1 D toric code:

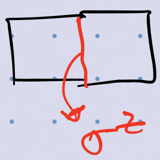
$H = - \sum_v$ 

$- \sum_\square$ 

- There exist two kinds of excitations.

i). 

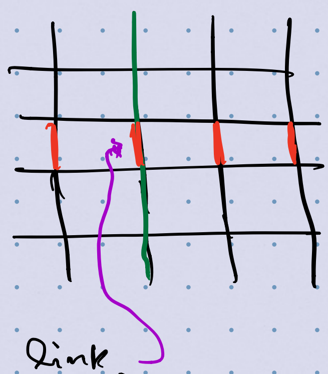
has two $+ = -1$ excitations at the ends of the link.

ii). 

creates two plaquette excitations.

we can take two $+ = -1$ excitations; take them around a non-contractible cycle and annihilate,

Or we could do the same with plaquette excitations.



However $\sigma^x \sigma^z = -\sigma^z \sigma^x$ at this link.

$$\Rightarrow \quad W_y T_x = - T_x W_y$$

We could do the same with the other cycles.

$$W_x T_y = -T_y W_x .$$

That is, 2 copies of Heisenberg algebra with 2 generators

$\Rightarrow$ $2^2 = 4$ ground states.

Ground states can not be distinguished by a local measurement $\Big\} \Leftrightarrow$ Topological order.

LEET in the continuum:

<u>Claim</u>: It is the $\mathbb{Z}_2$ gauge theory in 2+1 D.

To see the role that 2 in $\mathbb{Z}_2$ plays, I will study $\mathbb{Z}_N$ theory instead.

Of course $\mathbb{Z}_N$ gauge theory is a theory of $\mathbb{Z}_N$ bundles which does not have a spacetime dependent connection $A(x)$.

But Banks-Seiberg (1011.5120) have given a presentation of $\mathbb{Z}_N$ gauge theory in terms of connections on $U(1)$ bundles that high energy theorists (like me) are more used to.

$$\mathcal{L} = \frac{N}{2\pi} B\, dA$$

A, B — 1-form gauge fields (i.e. connections on $U(1)$ bundles).

$$\begin{pmatrix} A \to A + d\xi \\ B \to B + d\chi \end{pmatrix}$$

<u>Fact 1</u> : $N \in \mathbb{Z}$.

<u>Reason:</u> On a closed $M_3 = \partial M_4$,

$$\frac{N}{2\pi} \int_{M_3} B\, dA = \int_{M_4} \frac{N}{2\pi}\, dB \wedge dA$$

This is independent of choice of M_4 with

$$\frac{N}{2\pi} \int_{\overline{M}_4} dB \wedge dA \in 2\pi \mathbb{Z} \quad \text{for closed } \overline{M}_4.$$

We can consider the case $\overline{M}_4 = \Sigma_{g_2} \times \Sigma'_{g_2}$ and A, B such that $\oint_{\Sigma_{g_2}} dA \in 2\pi\mathbb{Z}$, $\oint_{\Sigma'_{g_2}} dB \in 2\pi\mathbb{Z}$

$$\Rightarrow \boxed{N \in \mathbb{Z}}$$

Fact 2: No local gauge-invariant operators.

By EOM, $dA = dB = 0$.

Fact 3: The theory has Wilson line observables.

$$U = e^{i\oint A}$$

$$V = e^{i\oint B}$$

However, $U^N = V^N = 1$.

Reason: We can compute the correlation function

$$\langle U^q(\gamma)\, V^r(\gamma') \rangle = e^{\frac{2\pi i}{N} q r \, lk(\gamma,\gamma')}$$

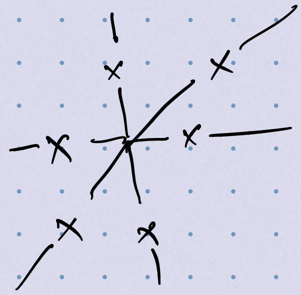
$q \to q + N$ OR $r \to r + N$ label the same operator.

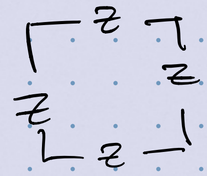
for $N = 2$, the two generating lines satisfy $UV = -VU$. This is the same commutation we saw above.

On T^2, we get two copies of this algebra.

$\Rightarrow$ 4 states.

3+1 D toric code:

$H = -\sum_{*}$ 

$-\sum_{\square}$ 

LEET in Continuum:

$\mathbb{Z}_2$ 3+1D gauge theory / BF theory.

For $\mathbb{Z}_N$, $\mathcal{L} = \frac{N}{2\pi} \, b \wedge da$

Now a : 1-form gauge field

b : 2-form gauge field

Gauge transformations : $a \to a + d\lambda$

$b \to b + d\mu$

Observables: Wilson lines : $e^{i\oint a}$

Wilson surfaces : $e^{i\oint b}$

④ Foliations, Foliated Gauge Fields and Foliated QFT of X-cube model:

(4.1). Foliations:

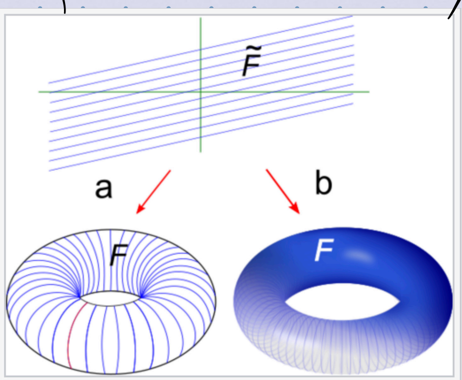
Def: A <u>k-dim foliation</u> of an n-dim manifold M is a decomposition of M into a union of disjoint k-dimensional submanifolds.

Each k-dimensional submanifold in a foliation is called a <u>leaf</u>.

Example: ① For $M = \mathbb{R}^4_{t,x,y,z}$, we can give a 3-dim foliation by fixing leaves to be the $z = \text{constant}$ slices.

② Take $M = \mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}$

then straight lines in $\mathbb{R}^2$ give leaves of a foliation on $\mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}$.



The linear foliation $\tilde{\mathcal{F}}$ on $\mathbf{R}^2$ passes to the foliation $\mathcal{F}$ on T^2. a) the slope is rational (linear foliation); b) the slope is irrational (Kronecker foliation).

Later, we will be mostly interested in codimension-1 foliations.

In particular, we will define a codimension-1 foliation through a nowhere zero 1-form field e.

At $p \in M$, the tangent space $T_p N$ of a leaf N is the kernel of e.

Frobenius theorem: This gives a foliation only if $e \wedge de = 0$.

In example ① above, $e = dz$. then $e \wedge de = 0$ trivially.

But e and γe for a scalar function γ give the same foliation.

"Godbillon-Vey invariant" of a foliation:

$$[\beta \wedge d\beta] \in H^3(M, \mathbb{R}) \longrightarrow (*)$$

where β is defined through $de = e \wedge \beta$.

(*) is independent of the choice of e and β. (math/0111137).

4.2 Foliated QFT of X-cube model :

- Slagle (2008.03852) discusses the following "foliated QFT".
- It lives on a 4-manifold M_4 equipped with n_f (transverse?) codimension-1 foliations.

- It depends on $N, M_k, n_k \in \mathbb{Z}$ and $k = 1, \ldots, n_f$.

  k labels k^{th} foliation defined by e^k such that $e^k \wedge de^k = 0$. (we are not summing over k here).

- It describes X-cube model when $M_4 = \mathbb{R}_+ \times M_3$,

  $n_f = 3$ transverse foliations on M_3,

  $N = 2$, $M_k = 2 \ \forall k$, $n_k = 1$

$$S = \int_{M_4} \underbrace{\frac{N}{2\pi} b \wedge da}_{\text{①}} + \sum_{k=1}^{n_f} \left(\underbrace{\frac{M_k}{2\pi} dB^k \wedge A^k}_{\text{②}} + \underbrace{\frac{n_k M_k}{2\pi} b \wedge A^k}_{\text{③}} \right)$$

① (3+1)-D $\mathbb{Z}_N$ Gauge theory

② "A stack of 2+1D $\mathbb{Z}_{M_k}$ theories"

③ Interaction between (2+1)-D and (3+1)-D gauge theories

We've already studied ①. It has observables $e^{i\oint a}$ & $e^{i\oint b}$ with no mobility constraints.

Let's focus on ② for now. (i.e. the case $n_K = N = 0$).

Later, we will study ① + ② + ③ to study the X-cube model.

② : $$S = \int_{M_4} \sum_{K=1}^{n_f} \frac{M_K}{2\pi} dB^K \wedge A^K$$

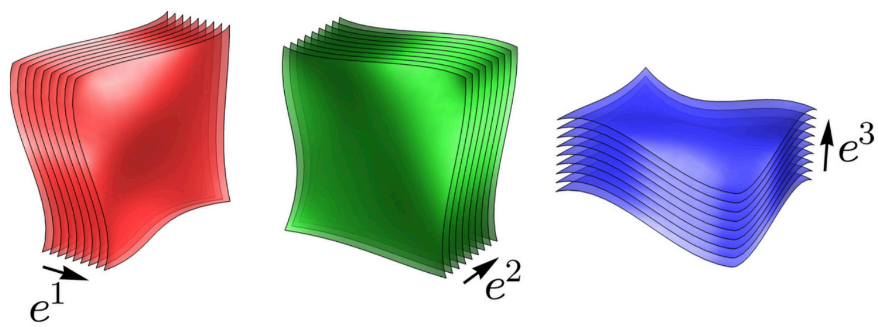


FIG. 1. A depiction of some leaves (colored surfaces) for three different foliations. A foliation consists of an infinite number of infinitesimally-spaced layers, which are called leaves.

For now, fix $n_f = 1$ to see how Physics depends on the choice of foliation.

$$S = \int_{M_4} \frac{M}{2\pi} dB \wedge A .$$

We want this action to describe a "stack" of $2+1$ D $\mathbb{Z}_M$ gauge theories.

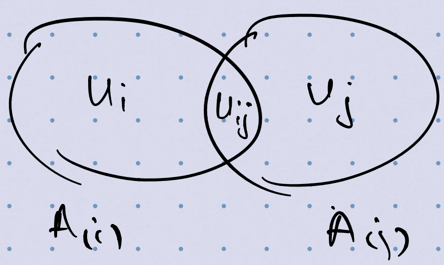
Slagle: Allow A to be defined independently on each leaf.

Question: why not do the same for B?

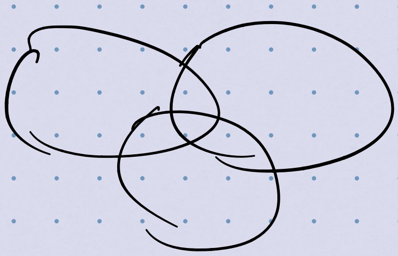
Formulation: Let A be a "(1+1)-form foliated gauge field"

i.e. it is a 2-form gauge field with some constraints

Recall A 2-form gauge field can be defined on a "good" open cover as follows.



$$A_{(i)} - A_{(j)} = dg_{ij}$$



$$g_{ij} + g_{jk} + g_{ki} = df_{ijk}$$

... and so on.

For a $(l+1)$-form foliated gauge field, constrain

$$A_{(i)} \wedge e = 0 \qquad \forall\,(i)$$

Since $A_{(i)} - A_{(j)} = dg_{ij}$,

also constrain $g_{ij} \wedge e = 0$

and so on ...

<u>Advantage?</u> It helps define discontinuous gauge fields.

Say $e = dz$ in some local chart.

Then $A \wedge dz = 0 \Rightarrow A = \tilde{A}(z) \wedge dz$

For a 2-form gauge field, dA is a 3-form on M.
That forbids discontinuities in A.

Here, dA does not involve a derivative in z.

Therefore $\tilde{A}(z)$ can be discontinuous in z.

- A can also have δ-functions, e.g.

$$A = \delta(z-z_0)\, dy \wedge dz.$$

- At a fixed z, A has only 3 components, A_{tz}, A_{xz}, A_{yz}

Gauge transformations of A?

Impose invariance under

$$A \to A + d\xi$$

for $\xi \wedge e = 0$.

Again, if $e = dz$, $\xi = \tilde{\xi}\, dz$

$$A_{\mu z} \longrightarrow A_{\mu z} + \partial_\mu \tilde{\xi}$$

This indicates that a $(1+1)$-form foliated gauge field is a tool to package a family of 1-form gauge fields. Can it be made more precise?

What about B-field?

Slagle: Let B be a 1-form gauge field on M_4 with a larger group of gauge transformations.

$$B \longrightarrow B + dX + \alpha \qquad \text{where } \alpha \wedge e = 0$$

So that $\alpha = \tilde{\alpha}\, dz$

* B_z is not physical. B has degrees of freedom of a 1-form gauge field on a leaf.

<u>Observables for foliated $\mathbb{Z}_M$ gauge theory:</u>

① No local observables as $dA = dB = 0$.

② Candidate line operator : $e^{i\oint B}$.

Is it gauge invariant?

$$e^{i\oint_\gamma B} \longrightarrow e^{i\oint_\gamma B + i\oint d\chi + i\oint \alpha}$$

$e^{i\oint_\gamma \alpha} \neq 1$ for an arbitrary 1-form α.

<u>But</u> $\alpha \wedge e = 0 \Rightarrow e^{i\oint_\gamma \alpha} = 1$ if γ lies in a leaf.

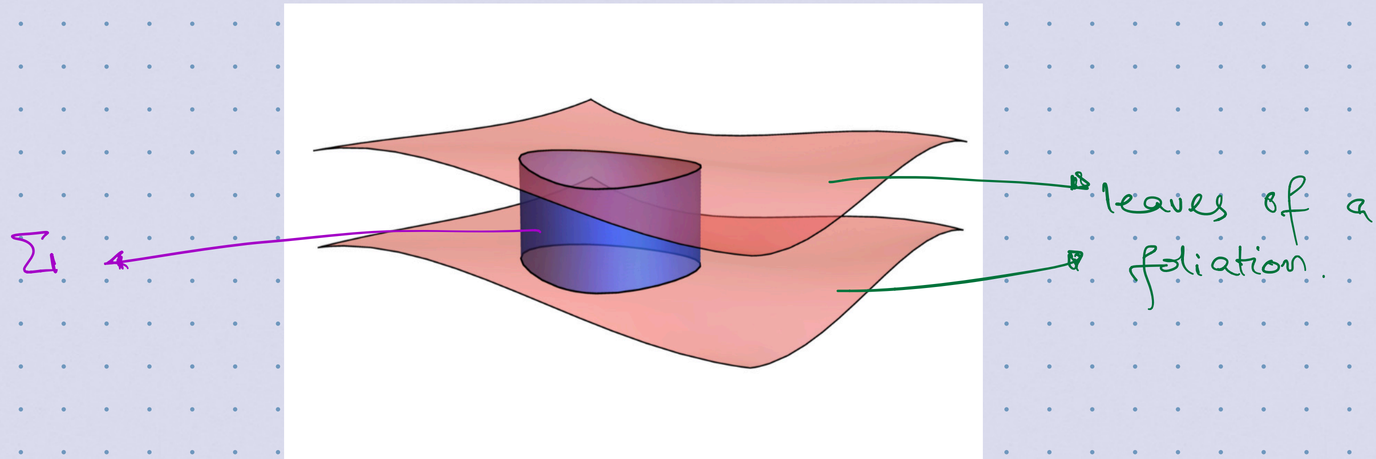
i.e, <u>Restricted mobility</u>!

③ Candidate Surface Operator : $e^{i\oint_{\Sigma_i} A}$

Is it gauge invariant?

$$A \to A + d\xi \qquad \text{where } \xi \wedge e = 0.$$

$\Rightarrow$ Σ_i must either be closed or have boundaries at leaves.



Summary: Gauge transformation parameters (α and ξ) that are foliated gauge fields explain mobility constraints

So far we discussed $\int_{M_4} \sum_{k=1}^{n_f} \frac{M_k}{2\pi} dB^k \wedge A^k$ for $n_f = 1$

For $n_f > 1$, we get n_f decoupled theories. The observables are $e^{i\oint B^k}$ and $e^{i\oint A^k}$ with mobility constraints dictated by k^{th} foliation.

In the Lagrangian of X-cube model, all A^k couple to b, i.e. theories will not be decoupled anymore.

Now we discuss the full theory.

$$S = \int_{M_4} \underbrace{\frac{N}{2\pi} b \wedge da}_{\substack{\text{(3+1)-D} \\ \mathbb{Z}_N \text{ Gauge Theory} \\ 1}} + \sum_{k=1}^{n_f} \left(\underbrace{\frac{M_k}{2\pi} dB^k \wedge A^k}_{\substack{\text{" A stack} \\ \text{of 2+1D} \\ \mathbb{Z}_{M_k}\text{-theories"} \\ 2}} + \underbrace{\frac{n_k M_k}{2\pi} b \wedge A^k}_{\substack{\text{Interaction} \\ \text{between (2+1)-D} \\ \text{and (3+1)-D} \\ \text{gauge theories} \\ 3}} \right)$$

Our gauge transformation rules so far:

$$A^k \to A^k + d\xi^k \qquad (\xi^k \wedge e^k = 0)$$

$$B^k \to B^k + d\chi^k + \alpha^k \qquad (\alpha^k \wedge e^k = 0)$$

$$a \to a + d\lambda$$

$$b \to b + d\mu .$$

But 3 is not invariant under $A^k \to A^k + d\xi^k$ and $b \to b + d\mu$.

Slagle: Modify the gauge transformation rules.

$$A^k \to A^k + d\xi^k \qquad (\xi^k \wedge e^k = 0)$$

$$B^k \to B^k + d\chi^k + \alpha^k - n_k \mu \qquad (\alpha^k \wedge e^k = 0)$$

$$a \to a + d\lambda - \sum_k \frac{n_k M_k}{N} \xi_k$$

$$b \to b + d\mu .$$

To compare with X-cube model, I will set $M_4 = S^1_+ \times M_3$, $n_f = 3$ foliations of M_3, $M_k = N = 2$ and $n_k = 1$.

Observables:

As before, these gauge transformations lead to mobility constraints.

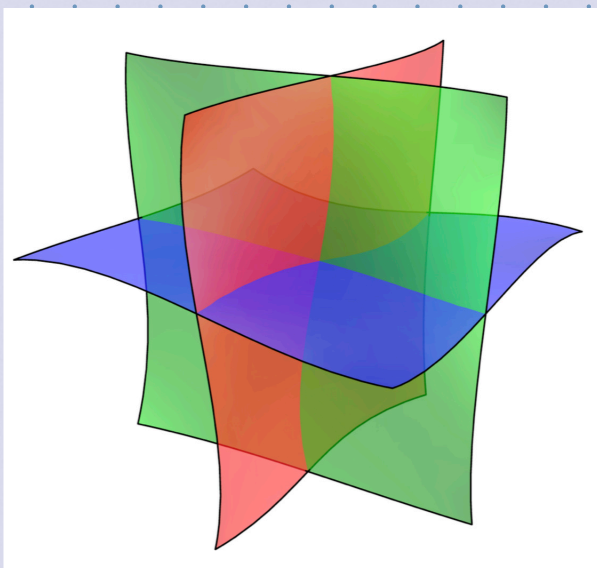
- $e^{i\oint b}$ is the Wilson surface of 3+1D $\mathbb{Z}_2$ gauge theory. We will discuss its interpretation in X-cube model below.

- $e^{i\int_\Sigma A^k}$ are the planons of X-cube model. (understand this better)..

- Claim: $e^{i\oint_\gamma a}$ is the world line of a fracton.

Under gauge transformation, $e^{i\oint_\gamma a} \to e^{i\oint_\gamma a - i\oint_\gamma \sum_k \xi_k}$

$\oint_\gamma \sum_k \xi_k = 0$ if γ is restricted to the intersection of three foliations.



- Claim: $e^{i\sum_k q_k \oint B^k}$ are lineons of X-cube model.

Invariance under $B^k \to B^k - n_k \mu \Rightarrow \boxed{\sum_k q_k n_k = 0}$

Here, $n_k = 1 \ \forall k \Rightarrow \boxed{q_1 + q_2 + q_3 = 0}$

Also, invariance under $B^k \to B^k + \alpha^k$ $\quad \boxed{\alpha^k \wedge e^k = 0}$

$\Rightarrow e^{i\sum_k q_k \oint B^k}$ restricted to a leaf of the k^{th} foliation if $q_k \neq 0$.

e.g. take $(q_1, q_2, q_3) = (0, 1, -1)$

$e^{i\phi B^2 - i\phi B^3}$ is the lineon along x-axis.

$(q_1, q_2, q_3) = (-1, 0, 1)$ is the lineon along y-axis.

And the sum of the two i.e. $(-1, 1, 0)$ is a lineon along z-axis.

(5) Some concluding remarks:

(5.1) Comparison with Seiberg–Shao ;

$$\mathcal{L}_{x\text{-cube}} = \frac{2}{2\pi}\left(\frac{1}{2} A_{ij} \hat{E}^{ij} + A_0 \hat{B}\right)$$

- (A_0, A_{ij}) and $(\hat{A}_0, \hat{A}_{ij})$ are $U(1)$ tensor gauge fields.
- $\mathcal{L}_{x\text{-cube}}$ is the $\mathbb{Z}_2$ tensor gauge theory.
- Seiberg–Shao also obtain mobility constraints as a consequence of gauge invariance.
- They also reproduce the ground state degeneracy, i.e. $2^{2L_x + 2L_y + 2L_z - 3}$ when they place their theory on T^4. Slagle has not done that but it is plausibly not too hard if the operator algebra of observables is identified.
- A guiding principle in their story is the presence of cubic spacetime symmetry. (as reviewed by Ryan two weeks ago). Slagle has taken a choice of foliation as a more fundamental ingredient. (Though on $\mathbb{R}^4$ with the standard foliation, the two backgrounds are the same).

- Slagle remarks that he has established in his conversations with Gorantla – Lam – Seiberg – Shao that on this background the two theories are dual to each other. (Is there a way to see it from path integral point of view?)

5.2 Dependence of Ground State Degeneracy on the choice of foliation:

- Any 3-manifold can be equipped with three transverse codimension-1 foliations. (Hardorp, 1980)
- GSD of X-cube model has been computed on various 3-manifolds through lattice calculations. (Shirley - Slagle - Wang - Chen, 1712.05892).

3-manifold	x-leaves	y-leaves	z-leaves	$\log_2$ GSD	c
T^3	$L_x \times T^2$	$L_y \times T^2$	$L_z \times T^2$	$2L_x + 2L_y + 2L_z - 3$	3
$S^2 \times S^1$	$L_x \times T^2$	$L_y \times T^2$	$L_z \times S^2$	$2L_x + 2L_y - 1$ $*$	1
S^3	$L_x \times S^2$	$L_y \times S^2$	$L_z \times S^2$	0	0
half-twist	$L_x \times T^2$	$L_y \times T^2$	$L_z \times T^2$	$2L_x + 2L_y + 2L_z$	0
half-twist	$(L_x - 1) \times T^2 + K^2$	$L_y \times T^2$	$L_z \times T^2$	$2L_x + 2L_y + 2L_z - 2$	2
half-twist	$(L_x - 1) \times T^2 + K^2$	$(L_y - 1) \times T^2 + K^2$	$L_z \times T^2$	$2L_x + 2L_y + 2L_z - 3$	3
$K^2 \times S^1$	$L_x \times T^2$	$L_y \times T^2$	$L_z \times K^2$	$2L_x + 2L_y + 2L_z - 2$	2
$\Sigma_g \times S^1$	$L_x \times T^2$	$L_y \times T^2$	$L_z \times \Sigma_g$	$2L_x + 2L_y + 2gL_z - 3g$	$3g$

Can these be reproduced through the foliated QFT we discussed?

5.3 Possible connections to mathematics:

- Can foliated gauge fields be given a more mathematical treatment?
- Is 'c' in $\log_2 GSD = b_x L_x + b_y L_y + b_z L_z - c$ an invariant of the total foliation of M_3?

5.4 New fracton models?

⋮

Any Questions?

And thank you!