Mirzakhani's "Simple Geodesics and Weil-Petersson volumes"

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Goal:

Present a recursive formula for the volume of $\mathcal{M}_{g,n}(L_1, \ldots, L_n) \sim$ the moduli space of genus g hyperbolic surfaces with n geodesic boundary components of lengths $L_i \in [0, \infty)$ when the Euler characteristic is negative.

vol($\mathcal{M}_{1,1}$)

Theorem (McShane's identity)

Let X be a hyperbolic once-punctured torus. Then

$$\sum_{\gamma}(1+\exp(\ell_\gamma(X)))^{-1} = 1/2$$

where the sum is over all simple closed geodesics (s.c.g.) γ on X.

Proposition

vol$(\mathcal{M}_{1,1}) = \pi^2/6$.

Proof.

- $\mathcal{M}_{1,1} = \mathcal{T}_{1,1}/\mathrm{Mod}_{1,1}$.
- $\mathcal{T}_{1,1}$ has nice description in terms of Fenchel-Nielsen coordinates:

$$\mathcal{T}_{1,1} \cong \{(\ell,\tau) :\ 0 < \ell, \tau \in \mathbb{R}\}$$

  depending on a s.c.c. α on $S = S_{1,1}$.
- If we had a fundamental domain for the action $\mathrm{Mod}_{1,1} \curvearrowright \mathcal{T}_{1,1}$ then vol$(\mathcal{M}_{1,1})$ equals the volume of the fund. domain and we're done.
- We won't use a fund. domain. Instead we find a finite measure on an intermediate cover $\mathcal{T}_{1,1}/\,\mathrm{Stab}(\alpha)$ that projects to the volume form.

vol($\mathcal{M}_{1,1}$)

Proof.

- Fix a s.c.c. α on $S = S_{1,1}$. Then

$$\begin{aligned} \mathcal{M}^*_{1,1} &:= \mathcal{T}_{1,1}/\,\mathrm{Stab}(\alpha) \\ &\cong \{(X,\gamma):\ X \in \mathcal{M}_{1,1}:\ \gamma \text{ a s.c.g. on } X\} \\ &\cong \{(\ell,\tau):\ 0 < \lambda, \tau \in \mathbb{R}\}/(\ell,\tau) \sim (\ell, \tau+\ell). \end{aligned}$$

- Let $f(x) = \frac{1}{1+e^x}$. So McShane's identity states $\sum_\gamma f(\ell_\gamma(X)) = 1/2$.
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$$\begin{aligned} \mathrm{vol}(\mathcal{M}_{1,1}) &= 2\,\mathrm{vol}(\mathcal{M}_{1,1}) \sum_\gamma f(\ell_\gamma(X)) \\ &= 2\int_{\mathcal{M}^*_{1,1}} f(\ell_\gamma(X))\,\mathrm{d}\,\mathrm{vol}_{\mathcal{M}^*_{1,1}}(X,\gamma) \\ &= 2\int_0^\infty \int_0^x f(x)\,\mathrm{d}y\mathrm{d}x = 2\int_0^\infty \frac{x}{1+e^x}\,\mathrm{d}x = \frac{\pi^2}{6}. \end{aligned}$$

□

$\mathrm{vol}(\mathcal{M}_{1,1})(L)$

Theorem (Generalized McShane's identity)

Let X be a hyperbolic one holed-torus with geodesic boundary of length L. Then

$$\sum_{\gamma} \mathcal{D}(L, \ell(\gamma), \ell(\gamma)) = L$$

where the sum is over all simple closed geodesics (s.c.g.) γ on X, not including the boundary and

$$\mathcal{D}(x, y, z) = 2 \log \left(\frac{e^{x/2} + e^{\frac{y+z}{2}}}{e^{-x/2} + e^{\frac{y+z}{2}}} \right).$$

Proposition

$\mathrm{vol}(\mathcal{M}_{1,1}(L)) = \frac{L^2}{24} + \pi^2/6.$

Proof.

- $\mathcal{M}_{1,1}(L) = \mathcal{T}_{1,1}(L)/\mathrm{Mod}_{1,1}$.
- $\mathcal{T}_{1,1}(L)$ has nice description in terms of Fenchel-Nielsen coordinates:

$$\mathcal{T}_{1,1}(L) \cong \{(\ell, \tau) : 0 < \lambda, \tau \in \mathbb{R}\}$$

depending on a s.c.c. α on $S = S_{1,1}$.

$\mathrm{vol}(\mathcal{M}_{1,1})(L)$

Proof.

- Fix a s.c.c. α on $S = S_{1,1}$. Then

$$\begin{aligned}\mathcal{M}^*_{1,1}(L) &:= \mathcal{T}_{1,1}(L)/\,\mathrm{Stab}(\alpha)\\ &\cong \{(X,\gamma):\ X \in \mathcal{M}_{1,1}(L):\ \gamma \text{ a s.c.g. on } X\}\\ &\cong \{(\ell,\tau):\ 0 < \lambda, \tau \in \mathbb{R}\}/(\ell,\tau) \sim (\ell, \tau + \ell).\end{aligned}$$

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$$\begin{aligned}L\,\mathrm{vol}(\mathcal{M}_{1,1}(L)) &= \mathrm{vol}(\mathcal{M}_{1,1}(L)) \sum_{\gamma} \mathcal{D}(L, \ell(\gamma), \ell(\gamma))\\ &= \int_{\mathcal{M}^*_{1,1}(L)} \mathcal{D}(L, \ell(\gamma), \ell(\gamma))\, \mathrm{d}\,\mathrm{vol}_{\mathcal{M}^*_{1,1}(L)}(X,\gamma)\\ &= \int_0^\infty \int_0^x \mathcal{D}(L,y,y)\, \mathrm{d}y\mathrm{d}x = \int_0^\infty x\mathcal{D}(L,x,x)\, \mathrm{d}x.\end{aligned}$$

□

$$\frac{\partial}{\partial x}\mathcal{D}(x,y,z) = H(y+z,x)$$

where $H(x,y) = \frac{1}{1+\exp\frac{x+y}{2}} + \frac{1}{1+\exp\frac{x-y}{2}}$.

Proof.

- Set $V(L) = \mathrm{vol}(\mathcal{M}_{1,1}(L))$.

$$LV(L) = \int_0^\infty x\mathcal{D}(L,x,x)\,\mathrm{d}x.$$

- Take $\frac{\partial}{\partial L}$ of both sides:

$$\begin{aligned}\frac{\partial}{\partial L}(LV(L)) &= \int_0^\infty x\frac{\partial}{\partial L}\mathcal{D}(L,x,x)\,\mathrm{d}x = \int_0^\infty xH(2x,L)\,\mathrm{d}x \\ &= \int_0^\infty \frac{x}{1+\exp\frac{2x+L}{2}} + \frac{x}{1+\exp\frac{2x-L}{2}}\,\mathrm{d}x\end{aligned}$$

- Set $y_1 = x + L/2$, $y_2 = x - L/2$ to get

$$\frac{\partial}{\partial L}(LV(L)) = \int_{L/2}^\infty \frac{y_1 - L/2}{1+e^{y_1}}\,\mathrm{d}y_1 + \int_{-L/2}^\infty \frac{y_2 + L/2}{1+e^{y_2}}\,\mathrm{d}y_2$$

$$= \left(\int_0^\infty \frac{y_1 - L/2}{1+e^{y_1}}\,\mathrm{d}y_1 - \int_0^{L/2} \frac{y_1 - L/2}{1+e^{y_1}}\,\mathrm{d}y_1\right) + \left(\int_0^\infty \frac{y_2 + L/2}{1+e^{y_2}}\,\mathrm{d}y_2 + \int_{-L/2}^0 \frac{y_2 + L/2}{1+e^{y_2}}\,\mathrm{d}y_2\right)$$

$$= 2\int_0^\infty \frac{x}{1+e^x}\,\mathrm{d}x - \int_0^{L/2} \frac{y_1 - L/2}{1+e^{y_1}}\,\mathrm{d}y_1 + \int_{-L/2}^0 \frac{y_2 + L/2}{1+e^{y_2}}\,\mathrm{d}y_2$$

$$= \pi^2/6 - \int_0^{L/2} (y - L/2)\left(\frac{1}{1+e^y} + \frac{1}{1+e^{-y}}\right)\,\mathrm{d}y = \pi^2/6 + L^2/8.$$

Proof.

- $\frac{\partial}{\partial L}(LV(L)) = \pi^2/6 + L^2/8.$
- $LV(L) = \int_0^L \pi^2/6 + x^2/8 \, \mathrm{d}x = L\pi^2/6 + L^3/24.$
- $V(L) = \mathrm{vol}(\mathcal{M}_{1,1}(L)) = \pi^2/6 + L^2/24.$

$\square$

$\mathrm{vol}(\mathcal{M}_{0,4})(L_1, L_2, L_3, L_4)$

Theorem (Generalized McShane's identity)

Let X be a hyperbolic four-holed sphere with geodesic boundaries of length L_1, L_2, L_3, L_4. Then

$$\sum_{i=2}^{4} \sum_{\gamma} \mathcal{R}(L_1, L_i, \ell_\gamma(X)) = L_1$$

where the sum is over all simple closed geodesics (s.c.g.) γ in the interior of X and

$$\mathcal{R}(x, y, z) = x - \log\left(\frac{\cosh(y/2) + \cosh\left(\frac{x+z}{2}\right)}{\cosh(y/2) + \cosh\left(\frac{x-z}{2}\right)}\right).$$

Proposition

$\mathrm{vol}(\mathcal{M}_{0,4}(L_1, L_2, L_3, L_4)) = \frac{1}{2}(4\pi^2 + L_1^2 + L_2^2 + L_3^2 + L_4^2)$.

Proof.

- $\mathcal{M}_{0,4}(\vec{L}) = \mathcal{T}_{0,4}(\vec{L})/\mathrm{Mod}_{0,4}$.
- $\mathcal{T}_{0,4}(\vec{L})$ has nice description in terms of Fenchel-Nielsen coordinates:

$$\mathcal{T}_{0,4}(\vec{L}) \cong \{(\ell, \tau) : \ 0 < \lambda, \tau \in \mathbb{R}\}$$

depending on a s.c.c. α on $S = S_{0,4}$

$\mathrm{vol}(\mathcal{M}_{0,4})(\vec{L})$

Proof.

- Fix a s.c.c. α on S (not homotopic into ∂S). Then

$$\begin{aligned}
\mathcal{M}^*_{0,4}(\vec{L}) &:= \mathcal{T}_{0,4}(\vec{L})/\operatorname{Stab}(\alpha)\\
&\cong \{(X,\gamma):\ X \in \mathcal{M}_{0,4}(\vec{L}):\ \gamma \text{ a s.c.g. on } X\}\\
&\cong \{(\ell,\tau):\ 0<\lambda,\tau\in\mathbb{R}\}/(\ell,\tau)\sim(\ell,\tau+\ell).
\end{aligned}$$

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$$\begin{aligned}
L_1 \operatorname{vol}(\mathcal{M}_{0,4}(\vec{L})) &= \operatorname{vol}(\mathcal{M}_{0,4}(\vec{L}))\sum_{i=2}^{4}\sum_{\gamma}\mathcal{R}(L_1,L_i,\ell_\gamma(X))\\
&= \int_{\mathcal{M}^*_{0,4}(\vec{L})}\sum_{i=2}^{4}\sum_{\gamma}\mathcal{R}(L_1,L_i,\ell_\gamma(X))\,\mathrm{d}\operatorname{vol}_{\mathcal{M}^*_{0,4}(\vec{L})}(X,\gamma)\\
&= \int_0^\infty\int_0^x\sum_{i=2}^{4}\mathcal{R}(L_1,L_i,x)\,\mathrm{d}y\mathrm{d}x\\
&= \int_0^\infty x\sum_{i=2}^{4}\mathcal{R}(L_1,L_i,x)\mathrm{d}x.
\end{aligned}$$

□

$$\frac{\partial}{\partial x}\mathcal{R}(x,y,z) = \frac{H(z,x+y)+H(z,x-y)}{2}$$

where $H(x,y) = \frac{1}{1+\exp\frac{x+y}{2}} + \frac{1}{1+\exp\frac{x-y}{2}}$.

Proof.

$$\begin{aligned}
L_1 \,\mathrm{vol}(\mathcal{M}_{0,4}(\vec{L})) &= \int_0^\infty x \sum_{i=2}^4 \mathcal{R}(L_1, L_i, x)\mathrm{d}x \\
\frac{\partial}{\partial L_1} L_1 \,\mathrm{vol}(\mathcal{M}_{0,4}(\vec{L})) &= \int_0^\infty x \sum_{i=2}^4 \frac{\partial}{\partial L_1}\mathcal{R}(L_1, L_i, x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty x \sum_{i=2}^4 H(x, L_1+L_i) + H(x, L_1 - L_i)\mathrm{d}x \\
&= \frac{1}{2}(4\pi^2 + 3L_1^2 + L_2^2 + L_3^2 + L_4^2). \\
L_1 \,\mathrm{vol}(\mathcal{M}_{0,4}(\vec{L})) &= \int_0^{L_1} \frac{1}{2}(4\pi^2 + 3L_1^2 + L_2^2 + L_3^2 + L_4^2)\mathrm{d}y \\
&= \frac{L_1}{2}(4\pi^2 + L_1^2 + L_2^2 + L_3^2 + L_4^2).
\end{aligned}$$

□

Overview of the general case

1. The generalized McShane's identity gives a formula for the length L_1 of a pre-specified boundary component β_1 of $X \in \mathcal{M}_{g,n}(L_1, \ldots, L_n)$ as a sum of terms over all pants containing β_1.
2. We integrate this formula over moduli space $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ using **Mirzakhani's integration formula** which gives us an integral in terms of volumes of moduli spaces of surfaces with smaller $|\chi|$:

$$L_1 \operatorname{vol}(\mathcal{M}_{g,n}(L_1, \ldots, L_n)) = \int_{\mathbb{R}^k_+} f(L, |\mathbf{x}|) V_{g,n}(\Gamma, \mathbf{x}, \beta, L) \, \mathbf{x} \cdot \mathrm{d}\mathbf{x}.$$

3. Although the integrand is explicit, it is difficult to integrate. Instead we differentiate with respect to L_1, integrate w.r.t. $\mathbf{x} \in \mathbb{R}^k_+$, then integrate with respect to L_1.

Generalized McShane's identity

Theorem

If X has boundary components $\beta_1, \ldots, \beta_n$ of lengths $L_1, \ldots, L_n$ then

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)) + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_\gamma(X)) = L_1$$

where

- $\mathcal{F}_1$ *is the set of all pairs* $\{\gamma_1, \gamma_2\}$ *bounding a pair of pants with* β_1*;*
- $\mathcal{F}_{1,i}$ *is the set of all* γ *bounding a pair of pants with* β_1 *and* β_i*;*
- $\mathcal{D}(x, y, z) = 2\log\left(\frac{e^{x/2}+e^{\frac{y+z}{2}}}{e^{-x/2}+e^{\frac{y+z}{2}}}\right)$;
- $\mathcal{R}(x, y, z) = x - \log\left(\frac{\cosh(y/2)+\cosh\left(\frac{x+z}{2}\right)}{\cosh(y/2)+\cosh\left(\frac{x-z}{2}\right)}\right)$.

Mirzakhani's Integration Formula

Theorem

Let

1. $\gamma = \sum_{i=1}^{k} c_i \gamma_i$ *be a multicurve on a surface* $S = S_{g,n}$*,*
2. $\ell_\gamma : \mathcal{M}_{g,n}(L) \to \mathbb{R}_+$ *is* $\ell_\gamma(X) = \sum_{i=1}^{k} c_i \ell_{\gamma_i}(X)$*,*
3. $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous,*
4. *define* $f_\gamma : \mathcal{M}_{g,n}(L) \to \mathbb{R}_+$*,* $f_\gamma(X) = \sum_{[\alpha] \in \mathrm{Mod}\cdot[\gamma]} f(\ell_\alpha(X))$*.*

Then

$$\int_{\mathcal{M}_{g,n}(L)} f_\gamma(X)\, dX = \frac{2^{-M(\gamma)}}{|\,\mathrm{sym}(\gamma)|} \int_{\mathbf{x} \in \mathbb{R}_+^k} f(|\mathbf{x}|) V_{g,n}(\Gamma, x, \beta, L)\ \mathbf{x} \cdot d\mathbf{x},$$

- $V_{g,n}(\Gamma, x, \beta, L) = \prod_{i=1}^{s} V_{g_i,n_i}(\ell_{A_i})$ *is the product of the volumes of the moduli spaces of surfaces obtained by cutting* $S_{g,n}$ *along* $\Gamma = (\gamma_1, \ldots, \gamma_k)$ *and fixing the lengths of* Γ *and the boundary curves.*
- $M(\gamma) = |\{i :\ \gamma_i$ *separates off a 1-handle from* $S_{g,n}\}|$*.*
- $\mathbf{x} \cdot d\mathbf{x} = x_1 \cdots x_k \cdot dx_1 \wedge \cdots \wedge dx_k$.

Goal: Prove the recursion Formula

Theorem

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) = \mathcal{A}^{con}_{g,n}(L_1, \hat{L}) + \mathcal{A}^{dcon}_{g,n}(L_1, \hat{L}) + \mathcal{B}_{g,n}(L_1, \hat{L})$$

where $\hat{L} = (L_2, \ldots, L_n)$, $\hat{L} \setminus L_j = (L_2, \cdots, \hat{L}_j, \ldots, L_n)$ *and*

$$\begin{aligned}
\mathcal{A}^{con}_{g,n}(L, \hat{L}) &= 2^{-1-m(g-1,n+1)} \int_0^\infty \int_0^\infty xyH(x+y, L_1) V_{g-1,n+1}(x, y, \hat{L})\, dxdy \\
\mathcal{A}^{dcon}_{g,n}(L, \hat{L}) &= 2^{-1} \int_0^\infty \int_0^\infty \sum_{a \in \mathcal{I}_{g,n}} xyH(x+y, L_1) V(a, x, y, \hat{L})\, dxdy \\
\mathcal{B}_{g,n}(x, L_1, \hat{L}) &= 2^{-1-m(g,n-1)} \int_0^\infty \sum_{j=2}^{n} x(H(x, L_1 + L_j) + H(x, L_1 - L_j)) V_{g,n-1}(x, \hat{L} \setminus \hat{L}_j)\, dx \\
H(x, y) &= \frac{1}{1 + \exp\frac{x+y}{2}} + \frac{1}{1 + \exp\frac{x-y}{2}} \\
m(g, n) &= \begin{cases} 1 & \mathit{if}\ (g, n) = (1, 1) \\ 0 & \mathit{otherwise} \end{cases}
\end{aligned}$$

$2\int_0^\infty \frac{x^{2i-1}}{1+e^x} = \zeta(2i)(2i-1)!(2 - 2^{-2i+2})$.

Goal: Prove the recursion Formula

$\mathcal{I}_{g,n}$ is the set of all ordered pairs $((g_1, I_1), (g_2, I_2))$ such that there is a pair of pants $P \subset S$ satisfying

- $\partial_P = \beta_1 \sqcup \gamma_1 \sqcup \gamma_2$ (with γ_1, γ_2 in the interior of S)
- $S \setminus P = S_1 \sqcup S_2$ where S_i is a surface of genus g_i and
- $\partial S_i = \gamma_i \cup \sqcup_{j \in I_i} \beta_j$.

If $a \in \mathcal{I}_{g,n}$ as above then

$$V(a, x, y, \hat{L}) = \frac{V_{g_1, n_1+1}(x, L_{I_1}) V_{g_2, n_2+1}(y, L_{I_2})}{2^{m(g_1, n_1+1)+m(g_2, n_2+1)}}.$$

McShane + Integration formula

1. Set

$$\begin{aligned}
\mathcal{D}(X) &= \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)) \\
\mathcal{R}_j(X) &= \sum_{\gamma\in\mathcal{F}_{1,j}} \mathcal{R}(L_1, L_j, \ell_\gamma(X)).
\end{aligned}$$

McShane $\Rightarrow \mathcal{D}(X) + \sum_{j=2}^n \mathcal{R}_j(X) = L_1$.

2. $L_1 V_{g,n}(L) = \int \mathcal{D}(X) + \sum_{j=2}^n \mathcal{R}_j(X)\; dX$.
3. Define

$$\mathcal{D}_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \mathcal{D}(X)\; dX$$

$$\mathcal{R}^j_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \mathcal{R}_j(X)\; dX.$$

4. Integration formula + transitivity of $\mathrm{Mod}_{g,n}$ on $\mathcal{F}_{1,j} \Rightarrow$

$$\begin{aligned}
\mathcal{R}^j_{g,n}(L) &= 2^{-m(g,n-1)} \int_0^\infty x\mathcal{R}(L_1, L_j, x) \cdot V(\mathcal{M}(S_{g,n}(\gamma_j), \ell_{\gamma_j} = x, L))\, \mathrm{d}x \\
&= 2^{-m(g,n-1)} \int_0^\infty x\mathcal{R}(L_1, L_j, x) \cdot V_{g,n-1}(x, L_2, \cdots, \hat{L}_j, \cdots, L_n)\, \mathrm{d}x.
\end{aligned}$$

How to compute $\frac{\partial}{\partial L_1} \sum_{j=2}^n \mathcal{R}^j_{g,n}(L)$.

$$\frac{\partial}{\partial x}\mathcal{R}(x,y,z) = \frac{H(z,x+y)+H(z,x-y)}{2}$$

where $H(x,y) = \frac{1}{1+\exp\frac{x+y}{2}} + \frac{1}{1+\exp\frac{x-y}{2}}$.

1. $\frac{\partial}{\partial L_1}\mathcal{R}^j_{g,n}(L) =$
   $2^{-m(g,n-1)} \int_0^\infty x(H(x,L_1-L_j)+H(x,L_1+L_j)) \cdot V_{g,n-1}(x,L_2,\cdots,\hat{L}_j,\cdots,L_n)\,\mathrm{d}x.$
2. So $\mathcal{B}_{g,n} = \frac{\partial}{\partial L_1}\sum_{j=2}^n \mathcal{R}^j_{g,n}(L) =$
   $2^{-m(g,n-1)-1}\sum_{j=2}^n \int_0^\infty x(H(x,L_1-L_j)+H(x,L_1+L_j)) \cdot V_{g,n-1}(x,L_2,\cdots,\hat{L}_j,\cdots,L_n)\,\mathrm{d}x.$

How to compute $\frac{\partial}{\partial L_1}\mathcal{D}_{g,n}(L)$.

$\mathrm{Mod}_{g,n}$ does not act transitively on $\mathcal{F}_1$. So first we decompose $\mathcal{F}_1$ into a disjoint union of $\mathrm{Mod}_{g,n}$ orbits.

Let γ_1, γ_2 be disjoint scc's such that $\beta_1, \gamma_1, \gamma_2$ bound a pair of pants P.

1. Let A^{con} be the set of all such $\{\gamma_1, \gamma_2\}$ such that $S \setminus P$ is connected.
2. $\mathrm{Mod}_{g,n}$ acts transitively on A^{con}.
3. For $a = ((g_1, I_1), (g_2, I_2)) \in \mathcal{I}_{g,n}$, let A_a be the set of all $\{\gamma_1, \gamma_2\}$ such that $S \setminus P = S_1 \sqcup S_2$ where S_i has genus g_i and $\partial S_i = \gamma_i \cup \sqcup_{j \in I_i} \beta_j$.
4. $\mathrm{Mod}_{g,n}$ acts transitively on A_a.
5. $\mathcal{F}_1 = A^{con} \cup \cup_{a \in \mathcal{I}_{g,n}} A_a$.
6. Recall

$$\mathcal{D}_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \mathcal{D}(X)\, dX = \int \sum_{\{\gamma_1,\gamma_2\} \in \mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))\, \mathrm{d}X.$$

How to compute $\frac{\partial}{\partial L_1}\mathcal{D}_{g,n}(L)$.

1. Recall

$$\mathcal{D}_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \mathcal{D}(X)\, dX = \int \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))\, \mathrm{d}X.$$

2. Integration formula $\Rightarrow$

$$\begin{aligned}&\int \sum_{\{\gamma_1,\gamma_2\}\in A^{con}} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))\, \mathrm{d}X \\ = \quad & \frac{1}{2^{m(g-1,n+1)}} \int xy\mathcal{D}(L_1, x, y) V_{g-1,n+1}(x, y, \hat{L})\mathrm{d}x\mathrm{d}y.\end{aligned}$$

3. Integration formula $\Rightarrow$

$$\begin{aligned}&\int \sum_{\{\gamma_1,\gamma_2\}\in A_a} \mathcal{D}(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))\, \mathrm{d}X \\ = \quad & \frac{1}{2^{m(g_1,n_1+1)+m(g_2,n_2+1)}} \int xy\mathcal{D}(L_1, x, y) V_{g_1,n_1+1}(x, \hat{L}_{I_1}) V_{g_2,n_2+1}(x, \hat{L}_{I_2})\mathrm{d}x\mathrm{d}y.\end{aligned}$$

How to compute $\frac{\partial}{\partial L_1}\mathcal{D}_{g,n}(L)$.

$$\frac{\partial}{\partial x}\mathcal{D}(x,y,z) = H(y+z,x).$$

1. Integration formula $\Rightarrow$

$$\int \sum_{\{\gamma_1,\gamma_2\}\in A^{con}} \mathcal{D}(L_1,\ell_{\gamma_1}(X),\ell_{\gamma_2}(X))\,\mathrm{d}X$$
$$= \frac{1}{2^{m(g-1,n+1)}}\int xy\mathcal{D}(L_1,x,y)V_{g-1,n+1}(x,y,\hat{L})\mathrm{d}x\mathrm{d}y.$$

2. Integration formula $\Rightarrow$

$$\int \sum_{\{\gamma_1,\gamma_2\}\in A_a} \mathcal{D}(L_1,\ell_{\gamma_1}(X),\ell_{\gamma_2}(X))\,\mathrm{d}X$$
$$= \frac{1}{2^{m(g_1,n_1+1)+m(g_2,n_2+1)}}\int xy\mathcal{D}(L_1,x,y)V_{g_1,n_1+1}(x,\hat{L}_{I_1})V_{g_2,n_2+1}(x,\hat{L}_{I_2})\mathrm{d}x\mathrm{d}y.$$

3. Add these terms together, take the partial derivative to obtain

$$\frac{\partial}{\partial L_1}\mathcal{D}_{g,n}(L) = \mathcal{A}^{con}_{g,n}(L) + \mathcal{A}^{dcon}_{g,n}(L).$$