Mirzakhani and Weil-Petersson volumes I

Wednesday, October 7, 2020 11:55 AM

<u>Motivation:</u> JT gravity

GR is difficult to understand as a quantum thry

$\rightsquigarrow$ simpler: lower dim

Let's try in 2d:

Naively, $I = \int_\Sigma d^2x \sqrt{g}\, R$

surface Riem metric Ricci scalar

However: Purely topological (Gauß-Bonnet)

$\to$ add a (real-valued) scalar ϕ

$$I = \frac{1}{\kappa} \int_\Sigma d^2x \sqrt{g} \cdot \phi \cdot (R+2) \qquad \text{JT gravity}$$

<u>Classical solution:</u> $R = -2$

$\rightsquigarrow$ Σ is hyperbolic Riemann surface

Feynman path integral:

$$Z_\Sigma = \frac{1}{\mathrm{vol}} \int D\phi\, D_g \exp\left(-\frac{1}{\kappa}\int d^2x \sqrt{g}(R+2)\phi\right)$$

volume of diffeomorphism

integrate over $i\phi$ $\rightsquigarrow$ $\delta(R+2)$

$\to$ integral localizes on $\mathcal{M}_g$ = moduli space of 2-mfld's w/ hyperbolic structure

$\rightsquigarrow$ $Z(\Sigma) = C^{\chi(\Sigma)} \int e^{\omega}$

$$\rightsquigarrow \quad Z(\Sigma) = C^{\chi(\Sigma)} \underbrace{\int_{\mathcal{M}_g} e^{\omega}}_{\mathrm{vol}(\mathcal{M}_g)}$$

structure

sympl. form on $\mathcal{M}_g$

$$\chi(\Sigma) = 2-2g$$

There is a similar story if Σ has bdry components of prescribed lengths

again: $\qquad Z(\Sigma_{g,\vec{b}}) \sim \mathrm{vol}(\mathcal{M}_{g,\vec{b}})$

(see e.g. Witten @WHCGP)

In the next few lectures, we will describe Mirzakhani's work on these sympl. volumes

2 ways: ① via sympl. reduction & intersection theory of tautological classes (Today + next time)

Mirzakhani '06

② via McShane identities & recursion (Charlie & Lewis)

The moduli space $\mathcal{M}_{g,n}$

Let $S = S_{g,n}$ be an oriented, smooth surface $\left\{ \begin{array}{l} \text{genus} = g \\ \#\text{ bdry comp.} = n \end{array} \right\}$

$$\chi(S) = 2-2g-n < 0$$

i.e. $g=0 \Rightarrow n \geq 3$

$g=1 \Rightarrow n \geq 1$

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$g=1 \Rightarrow n \geq 1$

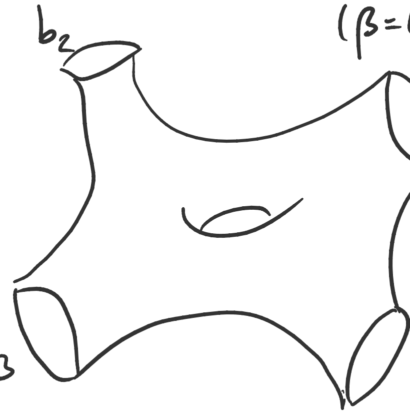
Def: The Teichmüller space $\mathcal{T}(S)$ is

$$\mathcal{T}(S) = \left\{ \begin{array}{l} \cdot\ X \text{ complete hyp. surface} \\ \cdot\ \text{diffeo } f: S \to X \end{array} \right\} \Big/ \sim$$

where $(X,f) \sim (Y,g)$ if $f \circ g^{-1}: Y \to X$ is isotopic to a conformal map

If $\partial S \neq \emptyset$ $(n \neq 0)$, Fix $\vec{b} = (b_i) \in \mathbb{R}_+^n$

$$\mathcal{T}(S,\vec{b}) = \left\{ \begin{array}{l} \cdot\ X \text{ complete hyp. surface} \\ \quad \text{w/ geodesic bdry} \\ \cdot\ f: S \xrightarrow{\sim} X \\ \quad \text{s.t. } \ell_\beta(X) = b_\beta \\ \quad (\beta = 1, \dots, n) \end{array} \right\} \Big/ \sim$$



We also write $\mathcal{T}_{g,\vec{b}} := \mathcal{T}_{g,n}(\vec{b}) := \mathcal{T}(S_{g,n}, \vec{b})$

The mapping class group

$$\mathrm{Mod}_{g,n} := \mathrm{Mod}(S_{g,n}) := \left\{ \begin{array}{l} \text{isotopy classes of orientation-pres.} \\ \text{homeo's } S \to S \text{ fix all bdry} \\ \text{comp. setwise} \end{array} \right\}$$

$\mathrm{Mod}_{g,n} \circlearrowright \mathcal{T}_{g,\vec{b}}$ by $\phi.(X,f) := (X, f\circ\phi)$

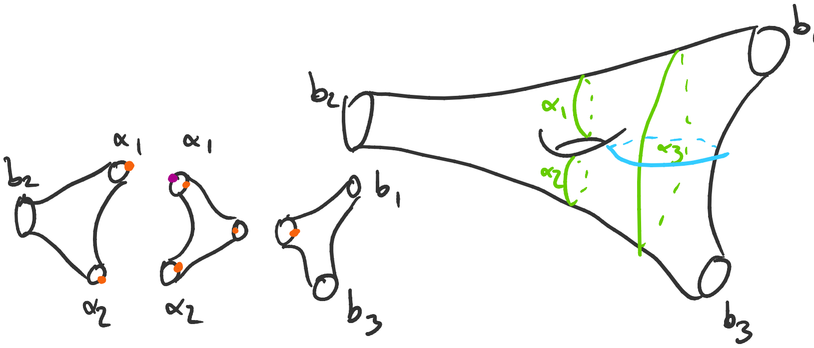
$\rightsquigarrow$ quotient $\mathcal{M}_{g,\vec{b}} := \mathcal{M}_{g,n}(\vec{b}) := \mathcal{M}(S_{g,n}, \vec{b}) = \mathcal{T}_{g,\vec{b}} / \mathrm{Mod}_{g,n}$

Moduli space of Riem. surfaces

Important features:

- $\mathcal{T}_{g,\vec{b}}$ has nice coordinates   Fenchel-Nielsen coordinates

  Fix a pair-of-pants decomp. $\mathcal{P}$ of $S_{g,n}$



$\rightsquigarrow$ get $3g-3+n$ curves

each of them has a unique geodesic representative α_i in its homotopy class

Fenchel-Nielsen coord's:

$$\{\underbrace{\ell_{\alpha_1}(X), \dots, \ell_{\alpha_k}(X)}_{\text{length of } \alpha_i}, \underbrace{\tau_{\alpha_1}(X), \dots, \tau_{\alpha_k}(X)}_{\text{twisting parameters}}\}$$

$\rightsquigarrow$ give an iso $\mathcal{T}_{g,\vec{b}} \xrightarrow{\sim} \mathbb{R}_+^k \times \mathbb{R}^k$

- $\mathcal{T}_{g,\vec{b}}$ has a natural sympl. form   Weil-Petersson form

  $\omega = \omega_{WP}$ (Goldman)

  that is $\mathrm{Mod}_{g,n}$-invariant

  $\rightsquigarrow$ $\mathcal{M}_{g,\vec{b}}$ has a sympl. form ω

  & we can compute $\mathrm{vol}(\mathcal{M}_{g,\vec{b}}) = \int e^{\omega}$

In FN-Coord's: $\omega = \sum^{k} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}$

In FN-coord's: $\omega_{WP} = \sum_{i=1}^{k} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}$

Symplectic reduction (Marsden-Weinstein reduction)

Let (M,ω) sympl. mfld $\circlearrowleft G$ cpct Lie gp preserves ω

Let $\rho: \mathfrak{g} \to \mathrm{Vect}(M)$ be the inf. action

Def: A moment map for the G-action on M is a G-equiv. map $\mu: M \to \mathfrak{g}^*$

s.t. $\forall X \in \mathfrak{g}$: $\iota_{\rho(X)}\omega = d\mu_X$

where $\mu_X = \mu \cdot X : M \to \mathbb{R}$

Rmk: Can always shift $\mu \to \mu + c$, $c \in \left(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\right)^*$

and any two moment maps differ by such a c

e.g.: · G semisimple $\rightsquigarrow$ moment unique

· G abelian $\rightsquigarrow$ $c \in \mathfrak{g}^*$

Ex: $(M,\omega) = \left(\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i\right)$, $\circlearrowleft G = U(1)$

has a moment map $\mu(\vec{x},\vec{y}) \cdot \partial_\alpha = -\frac{1}{2}\left(\|\vec{x}\|^2 + \|\vec{y}\|^2\right)$

Thm: Suppose (M,ω,G,μ) is a Hamiltonian G-space and suppose $G \circlearrowleft \mu^{-1}(0)$ freely.

Then: (i) $\mu^{-1}(0)$ ...

and suppose $G \curvearrowright \mu^{-1}(0)$ freely.

Then: ① $\mu^{-1}(0)/G$ is a manifold

② $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle

③ $\exists!$ sympl. form $\hat{\omega}$ on $\mu^{-1}(0)/G$ s.t.

$$\iota^*\omega = \pi^*\hat{\omega}$$

$$\begin{tikzcd} G \arrow[r] & \mu^{-1}(0) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & M \\ & \mu^{-1}(0)/G & \end{tikzcd}$$

<u>Ex:</u> $(M,\omega) = (\mathbb{C}^n, \omega_{std})$

$$\rightsquigarrow \mu^{-1}(0)/G \simeq (\mathbb{CP}^{n-1}, \omega_{FS})$$

<u>Q:</u> How do (M_a, ω_a) & (M_b, ω_b) relate for a "close to b"

<u>Thm:</u> [Normal Form Thm]

$G = U(1)^n$

Suppose $G \curvearrowright \mu^{-1}(0)$ freely. There exists an $\varepsilon > 0$ s.t. $G \curvearrowright \mu^{-1}(a)$ freely $\forall a \in B_\varepsilon(0)$

And we have: $M_a \underset{\text{diffeo}}{\simeq} M_0$

& if you pick a connection A on $\mu^{-1}(0) \to \mu^{-1}(0)/G$,

then $(M_a, \omega_a) \simeq (M_0, \omega_0 + a\Omega)$

symplectomorphic

curvature form of A

Rmk: The $U(1)^n$-bundle is really n $U(1)$-bundles $C_1, \dots, C_n$

$\rightsquigarrow$ Then the sympl. forms are related via

$$[\omega_a] = [\omega_0] + \sum_{i=1}^n a_i [\phi_i] \quad , \quad \phi_i \in c_1(C_i)$$

Cor: $(M, \omega, U(1)^n, \mu)$ a Hamiltonian system, $\mu^{-1}(0) \circlearrowleft U(1)^n$ freely

Then for sufficiently small $\|a\| < \varepsilon$, $\mathrm{vol}(M_a, \omega_a)$ is a polynomial of degree $d = \frac{\dim M_a}{2}$ given by

$$\mathrm{vol}(M_a, \omega_a) = \sum_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ |\alpha| \leq d}} C(\alpha)\, a^\alpha$$

where $$\alpha! \, (m - |\alpha|)! \; C(\alpha) = \int_{M_0} \phi_1^{\alpha_1} \cdots \phi_n^{\alpha_n} \omega^{d - |\alpha|}$$

Guillemin: Moment maps ...

'06 ... bordered $\mathbb{R}S$'s

[8] ?

Thm: (M_r, ω_r) sympl. $r = 1, 2$ $\dim = 2n$

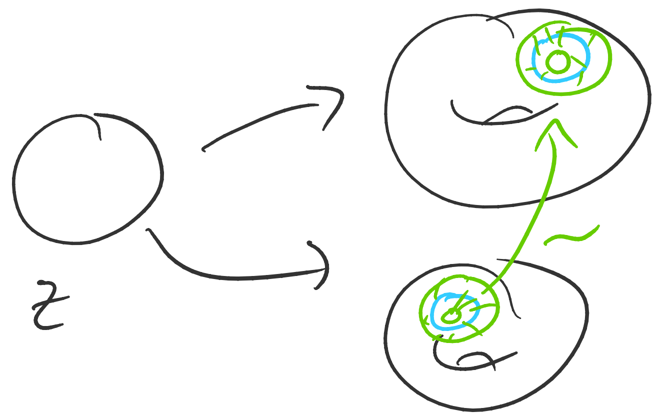
Z cpct. mfld $\dim k \geq n$ and $\iota_r : Z \to M_r$ coisotropic embeddings

s.t. $\iota_1^* \omega_1 = \iota_2^* \omega_2$

s.t. $\quad \iota_1^* \omega_1 = \iota_2^* \omega_2$

Then there exist nbhds U_i of $\iota_i(Z)$

& a sympl. $\phi : U_1 \longrightarrow U_2$



$$\omega = \pi^* \omega_0 + d(t \underset{\uparrow \text{cxn}}{A}) \quad , \quad -\varepsilon < t < \varepsilon$$