Jackiw-Teitelboim Gravity After Saad-Shenker-Stanford

11/2020 (errors or inaccuracies are due to YF)

Why JT gravity?

JT computes cool stuff in 2D, sort of like CS in 3D.

But really, why JT gravity?

We'd like to understand AdS/CFT. Why not start with $\mathrm{AdS}_2/\mathrm{CFT}_1$? AdS_2 is unlike $\mathrm{AdS}_{d>2}$ in that the backreaction from any excitation destroys the asymptotic AdS_2 geometry. So pure gravity in AdS_2 isn't so interesting. Moreover, in CFT_1,

$$T^{\mu}{}_{\mu} = 0 \implies H = 0 \tag{1}$$

(pure constraint), so conventional AdS/CFT only describes ground states.

But *nearly* AdS_2 gravity is described universally (to leading order) by **JT gravity**, a theory of 2D dilaton gravity whose bulk action is essentially

$$-\frac{1}{2}\int d^2x\sqrt{g}\phi(R+2). \tag{2}$$

The dilaton and metric are non-dynamical. The boundary modes are described universally by an action invariant under the global $SL(2,\mathbb{R})$ to which asymptotic diffs of AdS_2 are spontaneously broken (think Virasoro), given by a Schwarzian derivative.

Why the recent interest in the **$\mathbf{nAdS}_2/\mathbf{nCFT}_1$ correspondence**? One physical motivation is a toy model of holography (Kitaev, 2015):

$$\boxed{\text{SYK model}} \xrightarrow{\text{IR}} \boxed{\text{Schwarzian theory on } S^1} \xleftrightarrow{\text{holography}} \boxed{\text{JT gravity on } D^2} \tag{3}$$

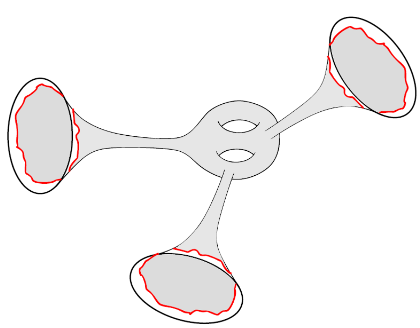
Another physical motivation is the universality of the low-energy physics of near-extremal black holes (which have an AdS_2 factor in their near-horizon geometry).

Main Results of SSS

The main result is the partition function of JT gravity

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_{\text{connected}} \tag{4}$$

on surfaces with n asymptotic (contrast with geodesic!) boundaries and arbitrary genus. **The genus expansion of this quantity coincides with that of a Hermitian matrix model.**



- **Physical Statement** (due to SSS):

$$\text{JT gravity} = \text{ensemble of random Hamiltonians.} \tag{5}$$

- **Mathematical Statement** (due to mathematicians):

$$\begin{gathered}\text{Mirzakhani's recursion relation for WP volumes} = \\ \text{Eynard-Orantin topological recursion for correlation functions in a matrix model.}\end{gathered} \tag{6}$$

This is an example of a very old connection between 2D gravity and random matrices.

Random Matrices

Consider a Hermitian (one-)matrix model:

$$\begin{gathered}\mathcal{Z} = \int dH\, e^{-L\,\mathrm{Tr}\,V(H)}, \\ H = L\times L \text{ Hermitian matrix ("Hamiltonian of boundary theory").}\end{gathered} \tag{7}$$

The basic observable is the thermal partition function

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}, \tag{8}$$

not to be confused with the partition function $\mathcal{Z}$ of the matrix model. Expectation values are

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle = \frac{1}{\mathcal{Z}}\int dH\, e^{-L\,\mathrm{Tr}\,V(H)} Z(\beta_1)\cdots Z(\beta_n). \tag{9}$$

These matrix models are solvable for $L\to\infty$. Correlation functions have a perturbative expansion (topological expansion) in $1/L^2$. For general $V(H)$, the "loop equations" (via topological recursion) determine all terms in this perturbative expansion in terms of a function

$$\rho_0(E) = \text{density of eigenvalues as } L\to\infty \text{ (suitably normalized).} \tag{10}$$

Example (Gaussian matrix model):

$$V(H) = \frac{H^2}{2} \implies \text{eigenvalue density} = L\rho_0(E), \quad \rho_0(E) = \frac{\sqrt{4-E^2}}{2\pi}, \quad \int_{-2}^{2}\rho_0(E)\, dE = 1 \tag{11}$$

(Wigner's semicircle law).

It is convenient to define the **resolvent**

$$R(E) = \mathrm{Tr}\,\frac{1}{E-H} = \sum_{j=1}^{L}\frac{1}{E-\lambda_j}, \quad E\in\mathbb{C}, \quad \{\lambda_j\} = \text{eigenvalues of } H. \tag{12}$$

For fixed H, this function is a sum of simple poles. We can also define the density of eigenvalues

$$\rho(E) = \sum_{j=1}^{L}\delta(E-\lambda_j). \tag{13}$$

For fixed H, this function is a sum of delta functions. After averaging over H, the poles in $R(E)$ are smeared into a branch cut, and $\rho(E)$ becomes a smooth function. The discontinuity of $R(E)$ across the real axis is given by

$$R(E+i\epsilon) - R(E-i\epsilon) = -2\pi i\rho(E). \tag{14}$$

For fixed H, this equality follows from

$$\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} = -\frac{2i\epsilon}{x^2+\epsilon^2} \xrightarrow{\epsilon\to 0} -2\pi i\delta(x). \tag{15}$$

It continues to hold after averaging over H. The thermal partition function $Z(\beta) = \operatorname{Tr} e^{-\beta H}$ is related to the resolvent $R(E)$ by an integral transform:

$$R(E) = -\int_0^\infty d\beta\, e^{\beta E} Z(\beta), \tag{16}$$

which makes sense for $E <$ ground-state energy $= 0$. The expression can be analytically continued in E after doing the integral.

Correlation functions of $R(E)$ have a $1/L$ expansion of the form

$$\langle R(E_1)\cdots R(E_n)\rangle_{\text{connected}} \simeq \sum_{g=0}^{\infty} \frac{R_{g,n}(E_1,\dots,E_n)}{L^{2g+n-2}}, \tag{17}$$

as do correlation functions of $Z(\beta)$. The parameter g corresponds to the genus of the diagrams that contribute to a given term, in 't Hooft's double-line notation. The connected part corresponds to subtracting one-point functions (connected geometries in gravity). The "$\simeq$" indicates an asymptotic series (nonperturbative effects matter).

The quantity $R_{0,1}$ ("disk") determines the leading density of eigenvalues $\rho_0(E)$, which is the seed for the topological recursion relations. As $L\to\infty$, $\rho(E)$ can be approximated by a smooth function that is the same for all typical matrices drawn from the ensemble. The unit-normalized density of eigenvalues in the large-L limit is

$$\rho_0(E) = \lim_{L\to\infty} \frac{1}{L}\langle \rho(E)\rangle. \tag{18}$$

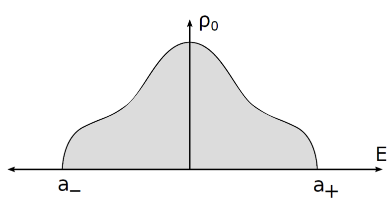
We focus on the simplest **one-cut matrix models**, with $\rho_0(E)$ supported on a single interval:

$$E \in [a_-, a_+], \qquad \int_{a_-}^{a_+} \rho_0(E)\, dE = 1. \tag{19}$$

We have

$$R_{0,1}(E+i\epsilon) - R_{0,1}(E-i\epsilon) = -2\pi i \rho_0(E) \Longleftrightarrow R_{0,1}(E) = \int_{a_-}^{a_+} d\lambda\, \frac{\rho_0(\lambda)}{E-\lambda} \tag{20}$$

(this is just the continuous version of the relation between the discrete $R(E)$ and $\rho(E)$).



We can determine ρ_0 or $R_{0,1}$ from $V(H)$ via a mean-field approximation (which becomes exact as $L\to\infty$). As usual, we can diagonalize H to write the matrix integral as an integral over eigenvalues. The Jacobian is a Vandermonde determinant:

$$\mathcal{Z} \propto \int d^L\lambda \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-L\sum_{j=1}^L V(\lambda_j)} = \int d^L\lambda\, e^{-\sum_{j=1}^L V_{\text{eff}}(\lambda_j)} \tag{21}$$

where

$$V_{\text{eff}}(\lambda_j) = LV(\lambda_j) - \sum_{i\neq j} \log[(\lambda_i - \lambda_j)^2] \approx LV(\lambda_j) - L \int d\lambda\, \rho_0(\lambda) \log[(\lambda - \lambda_j)^2]. \tag{22}$$

Each eigenvalue feels the same effective potential, and the equilibrium condition is

$$V'_{\text{eff}}(E) = 0 \implies V'(E) = 2 \int_{\text{PV}} d\lambda\, \frac{\rho_0(\lambda)}{E - \lambda}. \tag{23}$$

The reason for the principal value prescription is that we must omit the $i = j$ term from the sum in $V_{\text{eff}}(\lambda_j)$, so λ can never literally be E; we can achieve this by offsetting E from the real axis and taking the average of the results of doing so above and below:

$$V'(E) = R_{0,1}(E + i\epsilon) + R_{0,1}(E - i\epsilon). \tag{24}$$

One can use this relation to derive an explicit formula for $R_{0,1}$ in terms of V.

The **loop equations** (cf. Schwinger-Dyson equations) allow us to systematically compute all $R_{g,n}$. We start with

$$0 = \int d^L\lambda\, \frac{\partial}{\partial \lambda_a} \left[\frac{1}{E - \lambda_a} R(E_1) \cdots R(E_k) \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-L \sum_{j=1}^{L} V(\lambda_j)} \right] \tag{25}$$

and sum over a (to get expectation values of traces). By expanding in powers of $1/L$, we obtain a set of equations relating various $R_{g,n}$ that can be solved recursively.

Double-Scaled Matrix Models

JT gravity is actually dual to a matrix model with non-normalizable $\rho_0(E)$. This is a **double-scaled matrix model** where $L \to \infty$ while V is tuned such that $\rho_0(E)$ remains finite near the edge of the distribution.

The eigenvalue density of this matrix model has a single endpoint, which we take to be at $E = 0$. Hence it isn't normalizable (imagine taking one endpoint of the interval for a one-cut matrix model to ∞). This can be understood as a "double-scaled" limit of an ordinary matrix model. Specifically, we are interested in the leading density of eigenvalues

$$\rho_0^{\text{total}}(E) = \frac{e^{S_0}}{(2\pi)^2} \sinh(2\pi\sqrt{E}), \quad E > 0. \tag{26}$$

The "total" means that we are not dividing by L. To get this, imagine choosing a potential and a value of L such that

$$\rho_0^{\text{total}}(E) = \frac{e^{S_0}}{(2\pi)^2} \sinh\left(2\pi\sqrt{\frac{a^2 - E^2}{2a}}\right), \quad -a < E < a \tag{27}$$

for some finite a. To get the desired density, we shift $E \to E - a$ and take $a \to \infty$ such that $\rho_0^{\text{total}}(E)$ remains normalized to have integral L (this means that L/e^{S_0} becomes large in an a-dependent way). In the resulting double-scaled matrix model, the $1/L$ expansion is replaced by an e^{-S_0} expansion:

$$\langle R(E_1) \cdots R(E_n) \rangle_{\text{connected}} \simeq \sum_{g=0}^{\infty} \frac{R_{g,n}(E_1, \ldots, E_n)}{(e^{S_0})^{2g+n-2}}. \tag{28}$$

Likewise, we define the "unit-normalized" density of states as

$$\rho_0(E) = e^{-S_0} \rho_0^{\text{total}}(E) \tag{29}$$

rather than as $\rho_0 = \frac{1}{L}\rho_0^{\text{total}}$. The e^{-S_0} expansion is completely determined by ρ_0.

Topological Recursion

We can now state the Eynard(-Orantin) topological recursion for correlation functions of resolvents in our matrix model of interest, or specifically for the $R_{g,n}$. The point is that **topological recursion determines all $R_{g,n}$ (and hence all correlators), given the leading density of states $\rho_0(E)$.**

It turns out that the $R_{g,n}$ are generally double-valued functions of E. But they are single-valued functions of z where

$$z^2 = -E. \tag{30}$$

This is a coordinate on a double cover of the E-plane branched over the cut (in the double-scaled limit, branched over the origin). We also define the following single-valued function of z (simply a rescaled version of the leading density of states):

$$y(z) = \frac{\sin(2\pi z)}{4\pi}. \tag{31}$$

The locus $(z^2, y(z)) \subset \mathbb{C}^2$ is the "spectral curve" of the matrix model. It is a double cover of the complex E-plane, with the two sheets differing by $y \to -y$. Now we define

$$W_{g,n}(z_1, \ldots, z_n) = (-1)^n 2^n z_1 \cdots z_n R_{g,n}(-z_1^2, \ldots, -z_n^2), \tag{32}$$

except for the two special cases

$$W_{0,1}(z) = 2zy(z), \qquad W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}. \tag{33}$$

These are the base cases of the recursion relation, which determines all other $W_{g,n}$. To state it, let $J \equiv \{z_2, \ldots, z_n\}$. Then

$$\begin{aligned} &W_{g,n}(z_1, z_2, \ldots, z_n) = \text{residue at } z = 0 \text{ of} \\ &\frac{1}{4(z_1^2 - z^2)y(z)} \left[W_{g-1,n+1}(z, -z, J) + \sum_{\substack{I \cup I' = J \\ h + h' = g}}^{\prime} W_{h,1+|I|}(z, I) W_{h',1+|I'|}(-z, I') \right], \end{aligned} \tag{34}$$

where the sum excludes the cases $(I = J, h = g)$ and $(I' = J, h' = g)$.

The first few cases, for our spectral curve, are

$$\begin{gathered} W_{0,1} = \frac{z_1 \sin(2\pi z_1)}{2\pi}, \qquad W_{0,2} = \frac{1}{(z_1 - z_2)^2}, \qquad W_{0,3} = \frac{1}{z_1^2 z_2^2 z_3^2}, \\ W_{1,1} = \frac{3 + 2\pi^2 z_1^2}{24 z_1^4}, \qquad W_{1,2} = \frac{5(z_1^4 + z_2^4) + 3z_1^2 z_2^2 + 4\pi^2(z_1^4 z_2^2 + z_2^4 z_1^2) + 2\pi^4 z_1^4 z_2^4}{8 z_1^6 z_2^6}, \\ W_{2,1} = \frac{105}{128 z_1^{10}} + \frac{203\pi^2}{192 z_1^8} + \frac{139\pi^4}{192 z_1^6} + \frac{169\pi^6}{480 z_1^4} + \frac{29\pi^8}{192 z_1^2}. \end{gathered} \tag{35}$$

The key fact is that **the quantities $W_{g,n}$ are the Laplace transforms of the volumes $V_{g,n}(b_1, \ldots, b_n)$ of the moduli space of bordered Riemann surfaces**. Using Mirzakhani's recursion, the first few examples are computed to be

$$\begin{gathered} V_{0,1} = \text{undefined}, \qquad V_{0,2} = \text{undefined}, \qquad V_{0,3} = 1, \\ \widetilde{V}_{1,1} = \frac{b_1^2 + 4\pi^2}{48}, \qquad V_{1,2} = \frac{(4\pi^2 + b_1^2 + b_2^2)(12\pi^2 + b_1^2 + b_2^2)}{192}, \end{gathered} \tag{36}$$

$$V_{2,1} = \frac{(4\pi^2+b_1^2)(12\pi^2+b_1^2)(6960\pi^4+384\pi^2 b_1^2+5b_1^4)}{2211840},$$

where the tilde on $\widetilde{V}_{1,1}$ indicates that this moduli space has a $\mathbb{Z}_2$ symmetry, which we have quotiented by. We can check explicitly for these examples that

$$\boxed{W_{g,n}(z_1,\ldots,z_n) = \int_0^\infty b_1 db_1 e^{-b_1 z_1} \cdots \int_0^\infty b_n db_n e^{-b_n z_n} V_{g,n}(b_1,\ldots,b_n).} \tag{37}$$

This actually holds in general. In other words, Eynard-Orantin showed that the Laplace transform of Mirzakhani's recursion takes the form of topological recursion with the spectral curve $y = \sin(2\pi z)/4\pi$. This is a link

$$\text{loop equations of a double-scaled matrix model} \leftrightarrow \text{WP volumes.} \tag{38}$$

SSS use this statement to show that JT gravity is dual to a matrix ensemble.

Back to JT Gravity: Results

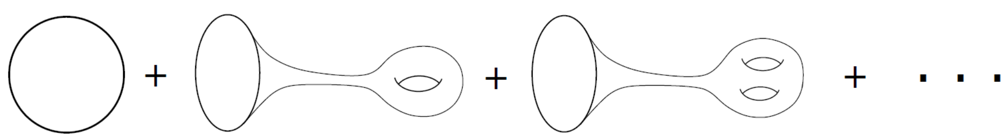
JT gravity has Euclidean action

$$I_{\text{JT}}[g_{\mu\nu},\phi] = -\frac{S_0}{2\pi}\left(\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right) - \frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2) - \int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1) \tag{39}$$

where:

- the first term evaluates to $-S_0\chi(\mathcal{M})$ (by Gauss-Bonnet) and weights topologies by $(e^{S_0})^\chi$ (where $\chi = 2-2g-n$ and, in BH language, S_0 is "extremal entropy"),
- the second term sets $R=-2$ (the "2" is conventional),
- the third term gives rise to the Schwarzian action on the boundary.

For instance, here are some topologies for $\langle Z(\beta)\rangle$ with weights $e^{(1-2g)S_0}$:



In nAdS$_2$/nCFT$_1$ (Maldacena-Stanford-Yang, 2016), we introduce a radial IR cutoff in AdS (i.e., a UV cutoff ϵ in the boundary theory). Correlation functions are computed by fixing the boundary lengths of $\mathcal{M}$ to be

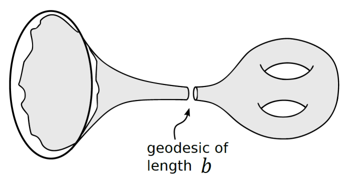
$$\frac{\beta_1}{\epsilon},\ldots,\frac{\beta_n}{\epsilon} \tag{40}$$

and imposing the boundary condition

$$\phi = \frac{\gamma}{\epsilon} \quad \text{(we take } \gamma = 1/2\text{)} \tag{41}$$

at each boundary. We take $\epsilon \to 0$ at the end.

To evaluate the path integral, we first integrate over ϕ, leaving an integral over surfaces of constant negative curvature. Since the boundary conditions only fix the lengths of the boundaries, we then have to integrate over their shapes (the "boundary wiggles"). The action for these wiggles (the Schwarzian theory) comes from the extrinsic curvature term. Within each topological class (for $g>0$), we also have to do a finite-dimensional integral over moduli.



To do these integrals, we imagine cutting the surface along minimal geodesics into a Riemann surface $\Sigma_{g,n}$ and n hyperbolic trumpets, where each trumpet has one Schwarzian (asymptotic) boundary and one geodesic boundary. Holding the lengths $b_1, \ldots, b_n$ of the geodesics fixed, we integrate over the moduli of $\Sigma_{g,n}$ and over the wiggles at the Schwarzian boundaries, and finally integrate over the b_i ("glue"). This gives

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_{\text{connected}} \simeq \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1,\ldots,\beta_n)}{(e^{S_0})^{2g+n-2}} \tag{42}$$

where

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \int_0^\infty b_1 db_1 \cdots \int_0^\infty b_n db_n V_{g,n}(b_1,\ldots,b_n) Z_{\text{Sch}}^{\text{trumpet}}(\beta_1,b_1)\cdots Z_{\text{Sch}}^{\text{trumpet}}(\beta_n,b_n) \tag{43}$$

and $V_{g,n}(b_1,\ldots,b_n)$ is the WP volume of the moduli space of bordered Riemann surfaces with genus g and n geodesic boundaries of lengths $b_1,\ldots,b_n$. The special cases are

$$Z_{0,1}(\beta) = Z_{\text{Sch}}^{\text{disk}}(\beta), \qquad Z_{0,2}(\beta_1,\beta_2) = \int_0^\infty b\, db\, Z_{\text{Sch}}^{\text{trumpet}}(\beta_1,b) Z_{\text{Sch}}^{\text{trumpet}}(\beta_2,b) \tag{44}$$

where

$$Z_{\text{Sch}}^{\text{disk}}(\beta) = \frac{e^{\pi^2/\beta}}{4\pi^{1/2}\beta^{3/2}}, \qquad Z_{\text{Sch}}^{\text{trumpet}}(\beta,b) = \frac{e^{-b^2/4\beta}}{2\pi^{1/2}\beta^{1/2}}. \tag{45}$$

This is an asymptotic series because $V_{g,n}$ grows as $(2g)!$. The matrix model gives a nonperturbative completion of this series; the nonperturbative effects are due to the dynamics of single eigenvalues (eigenbranes and probe branes).

Note that e^{-S_0} is nonperturbative in $G_N \sim 1/S_0$. These effects are perturbative from the matrix model POV but nonperturbative from the gravity POV. Nonperturbative effects in the matrix model are doubly nonperturbative in gravity.

The disk (genus zero) contribution to $\langle Z(\beta)\rangle$ is the Laplace transform of $e^{S_0}\rho_0(E)$. It is given by the partition function of the Schwarzian theory, and it leads to the density of states

$$\rho_0(E) = \frac{\sinh(2\pi\sqrt{E})}{(2\pi)^2}, \quad E > 0. \tag{46}$$

This follows from supersymmetric localization, among other methods (Stanford-Witten, 2017). One can then show that higher-genus contributions to arbitrary correlation functions satisfy the topological recursion relations with this density of states as input.

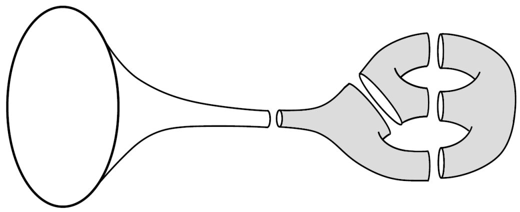
SSS show (using results of Eynard-Orantin) that to all orders in e^{-S_0}, JT gravity correlators $\langle Z(\beta_1)\cdots Z(\beta_n)\rangle$ coincide with correlators of $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$ for H drawn from a double-scaled matrix ensemble with Schwarzian density of states.

JT Gravity as Gauge Theory: Derivation

A term in the genus expansion of a connected correlator is given by the JT gravity path integral for a given topology with the S_0 term omitted from the action:

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \int d(\text{bulk moduli}) \int D(\text{boundary wiggles})\, e^{\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)}. \tag{47}$$

We need the measure for both integrals. For this, it is convenient to use the first-order formulation of JT gravity to rewrite it as a topological gauge theory in 2D. It turns out that the JT gravity path integral leads naturally to the WP measure on the bulk moduli.



Recall that the moduli space of bordered Riemann surfaces $\mathcal{M}_{g,n}(b_1,\ldots,b_n)$ is parametrized by the $2k$ Fenchel-Nielsen coordinates

- $\tilde{b}_1,\ldots,\tilde{b}_k$ (lengths of internal gluing boundaries),
- $\tau_1,\ldots,\tau_k$ (twist parameters),

where $k = 3g+n-3$. The WP symplectic form on this space is simply

$$\Omega = \sum_{i=1}^{k} d\tilde{b}_i \wedge d\tau_i. \tag{48}$$

The volume form is $\frac{1}{k!}\Omega^k$. Ω turns out to be invariant under change of pants decomposition, but to compute

$$V_{g,n}(b_1,\ldots,b_n) = \mathrm{vol}(\mathcal{M}_{g,n}(b_1,\ldots,b_n)), \tag{49}$$

one must restrict the integral to a fundamental domain.

Recall that on a Riemannian manifold, we can define an orthonormal frame by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab} \tag{50}$$

where $\omega^{ab}_\mu = -\omega^{ba}_\mu$ is chosen such that the vielbein e^a_μ is covariantly constant. Write $e^a = e^a_\mu\, dx^\mu$ and (in 2D) $\omega^{ab} = \epsilon^{ab}\omega$. The spin connection is determined by the no-torsion condition

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{51}$$

In 2D, we have

$$d^2x\,\sqrt{g} = e^1 \wedge e^2, \quad d^2x\,\sqrt{g}R = 2\,d\omega. \tag{52}$$

So in the first-order formulation, we have

$$\frac{1}{2}\int d^2x\,\sqrt{g}\phi(R+2) \to \int [\phi(d\omega + e^1\wedge e^2) + \phi_a(de^a + \epsilon^a{}_b\omega\wedge e^b)] = i\int \mathrm{Tr}(BF), \tag{53}$$

where we introduced Lagrange multipliers ϕ^a to enforce the no-torsion condition and wrote

$$B = -i\begin{pmatrix} -\phi^1 & \phi^2+\phi \\ \phi^2-\phi & \phi^1 \end{pmatrix}, \quad A = \frac{1}{2}\begin{pmatrix} -e^1 & e^2-\omega \\ e^2+\omega & e^1 \end{pmatrix}, \quad F = dA + A\wedge A. \tag{54}$$

This is an $SL(2,\mathbb{R})$ (more precisely, $\mathfrak{sl}(2,\mathbb{R})$) BF theory!

Integrating out B gives the constraint $F=0$, which reduces the path integral to an integral over flat connections. In BF theory language, the measure on the space of flat connections is very simple (Witten, 1991): it is induced by the symplectic form

$$\Omega(\sigma,\eta) = 2\int \mathrm{Tr}(\sigma\wedge\eta), \tag{55}$$

where σ, η are elements of the tangent space to the space of flat connections. This is a two-form in the sense that it takes two vectors in the tangent space to a point and gives a number. Modulo gauge transformations, the tangent space to A consists of gauge fields δA such that $A + \epsilon \delta A$ is flat to linear order in ϵ. One can show that Ω is gauge-invariant on the space of flat connections.

The key fact is that **this symplectic form on the space of flat $SL(2, \mathbb{R})$ connections is locally the same as the WP form on the moduli space of curves**. We can see this very concretely. Focus on the region near a particular gluing geodesic and choose coordinates ρ, x where ρ is transverse and x is longitudinal ($\rho < 0$ is one side of the tube, and $\rho > 0$ is the other). The metric is

$$ds^2 = d\rho^2 + \cosh^2(\rho)(b\, dx + \tau\delta(\rho)\, d\rho)^2 \equiv d\rho^2 + \cosh^2(\rho)\, dy^2 \tag{56}$$

where

$$x \sim x + 1, \quad y = bx + \tau\theta(\rho). \tag{57}$$

The metric is smooth in y but discontinuous in x: the two tubes have been glued together with a shift by τ in the x-direction. If we instead write this metric as an $SL(2, \mathbb{R})$ gauge field A and then consider small variations $\delta_i b, \delta_i \tau$ for $i = 1, 2$, then we compute that

$$\mathrm{Tr}(\delta_1 A \wedge \delta_2 A) = \frac{1}{2}(\delta_1 b\, \delta_2 \tau - \delta_2 b\, \delta_1 \tau)\delta(\rho)\, dx \wedge d\rho. \tag{58}$$

Integrating over ρ, x gives

$$\Omega(\delta_1 A, \delta_2 A) = \delta_1 b\, \delta_2 \tau - \delta_2 b\, \delta_1 \tau, \tag{59}$$

which is precisely the WP symplectic form.

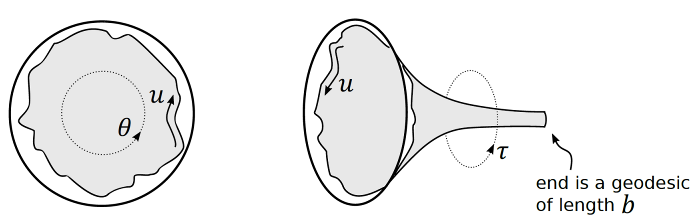
For physical applications, we want to impose asymptotically Euclidean AdS_2 boundary conditions. More precisely, to compute $Z(\beta)$ in JT gravity, we move the boundary in from infinity a little bit and impose (with $\gamma = 1/2$)

$$g_{uu}|_\partial = \frac{1}{\epsilon^2}, \quad \phi|_\partial = \frac{\gamma}{\epsilon}, \quad \epsilon \to 0, \tag{60}$$

where u is a proper length that runs from 0 to β, so that the total length is β/ϵ. These boundary conditions admit a "boundary graviton" (reparametrization) mode that allows the ϵ-regularized boundary to fluctuate while maintaining its length. In global coordinates

$$ds^2 = d\rho^2 + \sinh^2(\rho)\, d\theta^2 \tag{61}$$

for the hyperbolic disk (Euclidean AdS_2), this boundary is described by a function $\theta(u)$ (angle as a function of proper length); $\rho(u)$ is then fixed by the boundary conditions.



One can do the path integral over the boundary wiggles in the BF theory language to evaluate the partition function of the disk and the trumpet (the latter parametrized by the geodesic length b). E.g., for the disk, the JT gravity action reduces to the boundary extrinsic curvature term

$$I = \int_{\partial\mathcal{M}} \sqrt{h}\phi(K - 1) = -\frac{1}{2}\int du\, \mathrm{Sch}\left(\tan\frac{\theta(u)}{2}, u\right) = \frac{1}{4}\int_0^\beta du \left(\frac{\theta''^2}{\theta'^2} - \theta'^2\right). \tag{62}$$

This is the famous Schwarzian theory.

Now we have the ingredients to compute the genus-g partition function with n boundaries, $Z_{g,n}(\beta_1, \ldots, \beta_n)$. The gluing measure for a single trumpet follows from the WP form, and we must integrate over both the length b and the twist τ. Since twisting by b leaves the surface invariant, τ ranges from 0 to b, and we are left with a measure for the length parameter (which we integrate from 0 to ∞):

$$\int_\tau \Omega = db \int_0^b d\tau = b\, db. \tag{63}$$

The genus-zero density of states is obtained from

$$Z_{0,1}(\beta) = \int_0^\infty dE\, \rho_0(E) e^{-\beta E}, \tag{64}$$

which yields the desired spectral curve.

Using the relation between resolvents and thermal partition functions, we see that the $W_{g,n}$ are related to the $Z_{g,n}$ by an integral transform

$$W_{g,n}(z_1, \ldots, z_n) = 2^n z_1 \cdots z_n \int_0^\infty d\beta_1\, e^{-\beta_1 z_1^2} \cdots \int_0^\infty d\beta_n\, e^{-\beta_n z_n^2} Z_{g,n}(\beta_1, \ldots, \beta_n). \tag{65}$$

Using the gluing formula for $Z_{g,n}(\beta_1, \ldots, \beta_n)$ and substituting the explicit $Z_{\mathrm{Sch}}^{\mathrm{trumpet}}(\beta, b)$, we can perform the integral over β inside the integral over b. Then we get precisely the boxed relation between $W_{g,n}$ and $V_{g,n}$ that ensures, by Eynard-Orantin, that the $W_{g,n}$ satisfy the recursion relation with the desired spectral curve. So the sum over topologies in JT gravity reproduces the genus expansion of a double-scaled matrix model with the aforementioned spectral density.